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ABSTRACT

THIS IS ONE OF A SERIES OF UNITS INTENDED FOR BOTH PRESERVICE AND INSERVICE ELEMENTARY SCHOOL TEACHERS TO SATISFY A NEED FOR MATERIALS ON "NEW MATHEMATICS" PROGRAMS WHICH (1) ARE READABLE ON A SELF BASIS OR WITH MINIMAL INSTRUCTION, (2) SHOW THE PEDAGOGICAL OBJECTIVES AND USES OF SUCH MATHEMATICAL STRUCTURAL IDEAS AS THE FIELD AXIOMS, SETS, AND LOGIC, AND (3) RELATE MATHEMATICS TO THE "REAL WORLD," ITS APPLICATIONS, AND OTHER AREAS OF THE CURRICULUM. THIS UNIT EXPLORES WITH TEACHERS FUNDAMENTAL CONCEPTS INVOLVING NUMBER THEORY. CHAPTER 1 ANSWERS QUESTIONS CONCERNING THE MEANING OF THEORY OF NUMBERS AND WHY IT SHOULD BE STUDIED IN ELEMENTARY SCHOOL MATHEMATICS. CHAPTER 2 DEALS WITH VARIOUS SUBSETS OF THE WHOLE NUMBERS SUCH AS EVEN NUMBERS, ODD NUMBERS, PRIME NUMBERS, COMPOSITE NUMBERS, AND FACTORIZATION. CHAPTER 3 PRESENTS ELEMENTARY NOTIONS ON THE GREATEST COMMON DIVISOR AND LEAST COMMON MULTIPLE. CHAPTER 4 INTRODUCES TO THE ELEMENTARY TEACHER SOME ENRICHMENT TOPICS FROM NUMBER THEORY. (RP)

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An Introduction to the Theory of Numbers for Elementary School Teachers

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University of Michigan U. S. O. F. Project

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Unit I. INTRODUCTION

1 1 What Is the Theory of Numbers?

The theory of numbers has been characterized variously by different writers as: a descendant of Greek "arithmetica", number recreations and puzzles which interest students of higher mathematics, the purest branch of mathematics, the least applicable of all mathematics, one of the oldest branches of mathematics, and the most difficult of all mathematical disciplines. It is all of these and none of these, depending on your viewpoint It is an offspring from Greek arithmetica yet today's number theory bears little resenblance to the number worship of the ancient Greeks. The theory of numbers is more than an idle pastime such as recreations and puzzles might suggest. Whether or not the theory of numbers is considered the most pure or least applicable of mathematics depends on whether you are a number theorist or not. Number theory is certainly one of the oldest branches of mathematics.

We cannot allow the last characterization of number theory to go unqualified without immediately having the question raised, "Then why expect elementary teachers to study number theory?" It is not expected that elementary teachers study the type of number theory characterized by "the most difficult of mathematical disciplines." There are various levels of sophistication in this field just as there are in algebra, geometry and other fields of mathematics. This booklet will be such that any elementary

teacher with a knowledge of arithmetic and an acquaintance with set language will be able to follow the discussions.

We still have not answered the question, "What is number theory?" Perhaps the best way to answer it is to study some and then reflect on the question again. Explanations of unknown things have little meaning to a person before that person has some experience with those unknown things We now proceed on that assumption, however, we remark that this branch of mathematics confines itself to the properties of the set of integers--the positive and negative whole numbers.

1.2. Why Should Teachers Study the Theory of Numbers?

The most obvious answer to this question is that many topics from the theory of numbers are finding their way into the school curriculum at all levels. The emphasis in present day school mathematics is on understanding why as well as knowing how and what. Most everyone can remember some teacher who responded to "Why?" questions with "That is just the way you do it, don't ask me 'Why?'" Hopefully, such dogmatic teaching is rapidly becoming a thing of the past. The elementary teacher should know the "why" as well as the "how" and "what" so that he or she is able to lead students to an understanding of the why. A good grasp of the fundamental structure and nature of mathematics is necessary for knowing why, and the theory of numbers contributes to this understanding.

Another reason for having elementary teachers study number theory in some depth is so that they will know more than they expect their students to learn. This knowledge will contribute to the teacher's confidence. This confidence will in turn contribute to a healthier attitude toward mathematics in the classroom, and the students will be benefited. If the teacher knows more than his students, he will have insight into what they are going to meet in future mathematical studies and will realize the importance of what he teaches now.

Number theory also offers many interesting sidelights to mathematics which work nicely as enrichment materials. The history of mathematics contains many interesting anecdotes about number theory which can be used for motivating students in mathematics. Teachers with many of these anecdotes in their repertoire are more effective as teachers of mathematics.

Finally, teachers cannot really appreciate the subject of mathematics unless they have an opportunity to see as many facets of its as possible. They should see the many relationships between and the common elements of the various branches of mathematics. An appreciation of the structure of mathematics stemming from this insight will be reflected in more fruitful educational experiences for mathematics students.

1.3. Why Should Number Theory Be Studied in Elementary School Mathematics?

Most of the number theory that is included in elementary texts today is included in an incidental way. It is usually introduced in order that it may

be used in working with some more traditional topics such as addition of fractions. A section will generally not be devoted to the study of number theory alone, though there are some exceptions where topics in number theory are included as enrichment material and are not used in any other way.

It is unfortunate that more number theory is not included in the elementary school mathematics curriculum. It can add much to the understanding that youngsters gain of the nature of mathematics and to their appreciation of and attitude toward mathematics.

One problem facing all elementary teachers is how to get enough drill and review in fundamentals into their teaching. There is always the danger that drill will become meaningless and destroy initiative if students are assigned page after page of problems to give them practice in fundamentals. Number theory offers a nice solution to this dilemma. It is a good source of "incidental" drill material which focuses attention not on drill but on some interesting and new areas of mathematics. So, using number theory, the students can get the practice they need but in a painless manner. Also, the theory of numbers provides some concrete applications of whole numbers and students will be able to apply their skills to discovering some new properties of whole numbers.

Another ever present problem faced by elementary teachers is to motivate students and to interest them in mathematics. The necessity of

learning addition and multiplication facts and the algorithms of arithmetic offers a real challenge to the teacher to keep students' interest. Again the theory of numbers has some answers. The use of the history of mathematics as enrichment has been mentioned previously. Many other topics from the theory of numbers offer interesting and challenging sidelights to the regular mathematics curriculum. It should not be inferred that such enrichment is applicable only to the case of the capable student who has finished his assignment early and needs something to occupy him. Enrichment is possibly even more valuable for the slower student. Number theory has much to offer both of these students as it can offer a challenge to the former and yield problems easily understood by the latter. Such problems deal with whole numbers, even numbers, odd numbers, and the four fundamental arithmetic operations. Commonly, enrichment takes the form of allowing a capable student to move into material studied in the following grades. If this is not desirable, the theory of numbers offers an alternative. Many topics which can be selected from this field are found nowhere else in the elementary school program.

Modern mathematics programs place an emphasis on helping students to learn the structure of mathematics. Along with this, in the later elementary years students are exposed to some form of mathematical proof. This is a very difficult concept for teachers as well as students to grasp. Often "proofs" are given of statements which are obviously true and which the students have accepted long before. For this reason there is little understanding

developed of the nature of proof and its place in mathematics. The theory of numbers is very fruitful in offering opportunities for students to develop ideas of inductive and deductive reasoning. In fact, they can formulate their own conjectures (guesses) about relationships between numbers and with practice make "proofs" of them. Some examples of this are shown later.

For teaching inductive and deductive methods of proof in the classroom, there are some important points to be remembered. Whenever possible, ideas about what is to be proved and how to do it should come from the students. They must be allowed to try and to err and to say what they think. More mathematics is probably learned by working problems wrong (and having them corrected) than by working problems correctly. Creativity should be encouraged and a student should not be stopped if he goes off on a tangent that appears fruitful (and maybe some which do not). He will learn something about mathematics even if he goes up a blind alley.

Rigorous formal arguments cannot be expected from elementary students. A mediocre proof by a student is much more valuable than an elegant one done by the teacher. However, a gross error should not be overlooked and allowed to multiply itself later. If the essential ideas are presented by a student's "proof", then it should be accepted as correct.

The theory of numbers should give students some good ideas as to how mathematicians work. In it they can experience discovery, intuition, inductive and deductive reasoning and formulating (and holding to) definitions. Also, one pitfall of mathematical reasoning can be shown vividly by topics

from number theory. This pitfall is the tendency of accepting a statement as true after checking it in only a few specific cases. Some examples of this will be shown later.

It has been suggested here that the theory of numbers offers some promise as a fruitful topic to be included in the elementary curriculum You cannot really judge the value of number theory until you know something about it. This is the purpose of these units. You will find that most of the following topics are not new to you, but you will get a fresh look at them from another viewpoint. We now make some general comments about what to expect in the following units and then proceed to the theory of numbers.

1.4. What Is Assumed and What to Expect

Very little indication has been given yet of what numbers we deal with in the theory of numbers. In general, the field of number theory deals with the set of integers. This includes the counting numbers, zero and the negative integers. That is, the set of integers is the set

$$\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}\ .$$

The three dots at each end means that this set is infinite and that we imagine it continued on in the same fashion in both directions.

In our work here we will be concerned with a subset of the set of integers. We will speak only of the set of whole numbers. This includes the counting

numbers and zero. The set of whole numbers, then, is

$$\{0,1,2,3,\dots\}\ .$$

Note that it is also an infinite set. We will deal with various subsets of this set; for example, the sets of even numbers, of odd numbers, of prime numbers and of composite numbers. There is an important subset which we will not have occasion to deal with but will mention in passing here. This is the set of natural numbers

$$\{1,2,3,\dots\}\ .$$

Commonly, this set is called the set of counting numbers as was done above. This is a very important set for more formal studies of number theory than we will do here.

A developmental approach will be used throughout the units. This is done because it is probably the most fruitful. Also, it will give you an example of the type of approach recommended often today for teaching elementary school mathematics. Students are led to discover patterns and generalizations rather than being told them. It is hoped (and it appears so) that this leads to better understanding of mathematical concepts.

There are some assumptions made as to previous experience of the reader. The basic concepts and language of sets are assumed. Such terms as subset, union of sets, intersection of sets and disjoint sets are used. A knowledge of the basic principles governing the arithmetic operations is

also assumed. These principles are: associativity, commutativity, closure and the existence of identity elements for addition and multiplication, and the fact that multiplication distributes over addition. These principles are not necessary to the development but they are mentioned whenever they are used.

necessary to the development but they are mentioned whenever they are used.

Unit 2. PRIMES AND FACTORS

2.b Evenness and Oddness

Of the many possible divisions of the whole numbers into sets, disjoint or otherwise, we will first consider two, the set of even and the set of odd numbers. When a child first encounters the theory of numbers in elementary arithmetic, it is probably in the study of some of the basic properties of even and odd numbers. Historically, the study of even and odd numbers was the source of what we today call the theory of numbers. The many interesting properties of these numbers were studied by the Pythagoreans, a mystical organization in ancient Greece devoted to the study of numbers. Many of the ideas to be developed later in this unit were known by these early mathematicians.

Many uses are made of odd and even numbers today which are not familiar to many people. For example, odd and even numbers are used in numbering highways in the United States in such a way that north-south roads are named with odd numbers and east-west roads with even numbers.

numbers. We have U.S. 40 from east to west across the United States, I-94 from Detroit to Chicago and I-75 north-south. Most everyone is familiar with the numbering of houses, the odd numbers on one side of the street, the even on the other. A fact many may not know is that airline companies use even and odd numbers to help prevent head-on crashes between airplanes. This is done by having the flights in one direction at odd numbered thousands (e.g., 17,000) of feet of altitude and the flight in the other at even numbered thousands of feet of altitude. Such facts as these are interesting and should be good motivating devices for developing interest of students in mathematics. We now turn to the development of some mathematical concepts associated with the sets of even and odd numbers. Some questions in this area lend themselves very well to discovery and creativity. Some such questions are about sums and products of even and odd numbers. We shall consider this after general developments about even and odd numbers themselves.

Even Numbers

If we were to pose the question to someone, "What is an even number?" the answer might range from, "It's counting by twos," to "You can divide it by two," to "It ends in 0, 2, 4, 6, or 8" Such imprecise or vague notions will not suffice for our work and we shall attempt to develop more precise ideas. Consider the following:

$$0 = 2 \times 0$$
$$2 = 2 \times 1$$
$$4 = 2 \times 2$$
$$6 = 2 \times 3$$
$$8 = 2 \times 4$$

Do you see a pattern developing here? On the left we have the even numbers and on the right other names for these numbers all showing 2 x (some whole number). Suppose we continue the list of 2 x (some whole number) on the right:

$$? = 2 \times 5$$
$$? = 2 \times 6$$
$$? = 2 \times 7$$
$$\vdots$$
$$? = 2 \times 91$$
$$\vdots$$

Would we continue to get even numbers on the left? It appears so. Would we get "all" of the even numbers on the left? The "definitions" given above do not sufffice to answer these questions in a direct manner and we must formulate a more precise, workable definition Suppose we define "even number" in light of the pattern seen above.

Definition 1. An even number is any number which is the product of the number two and a whole number.

This fits the pattern observed before. It also agrees with our imprecise "definitions" given before. Does it agree with your concept of "even number"?

Suppose we agree to call this our official definition. It might appear that the use of two in this definition is rather arbitrary. Why couldn't we have used 3 or 4 or some other number? Suppose we investigate a pattern of products for other numbers as we did above for 2 and see what happens.

$3 \times 0 = 0$	$4 \times 0 = 0$	$6 \times 0 = 0$
$3 \times 1 = 3$	$4 \times 1 = 4$	$6 \times 1 = 6$
$3 \times 2 = 6$	$4 \times 2 = 8$	$6 \times 2 = 12$
$3 \times 3 = 9$	$4 \times 3 = 12$	$6 \times 3 = 18$
$3 \times 4 = 12$	$4 \times 4 = 16$	$6 \times 4 = 24$
⋮	⋮	⋮

Now if we tried to define even number in terms of 3 we would run into a little difficulty since some products of 3 are odd. Suppose we throw out the products that are odd. This leaves

$3 \times 0 = 0$

$3 \times 2 = 6$

$3 \times 4 = 12$.

⋮

Thought Exercise

All of the numbers listed avove are even What would be wrong with defining even number in terms of 3 using just these products? Would we get <u>all</u> even numbers? Staying in the set of whole numbers, how could we get 3 x (some whole number) = 2 ? Or 4 ? Or 8 ?

Suppose we consider 4 in the same way. You will note that the products of 4 x (any whole number) are all even so at least we don't need to discard any. Let's ask the same type of questions about 4 as we did about 3.

Thought Exercise:

Staying in the set of whole numbers $W = \{0,1,2,\dots\}$, can we find some number n such that $4 \times n = 2$? 6 ? 10 ? Consider the products of the form 6 x (some whole number) Could 6 be used to define even number so that we could be assured that our definition included all whole numbers? What about 2 ? 38 ?

Since it appears that use of 3, 4, or 6 in a definition of even number will not give us all even numbers, maybe we ought to question whether 2 does.

Thought Exercise:

Is $2 \times n$ always an even number whenever $n \in W$ (n is an element of the set of whole numbers)? Look at Definition 1 to help you answer. Suppose we ask the question: If m is an even number can we write $m = 2 \times n$ where n is some whole number? That is, is every even number of the form $2 \times n$? This is really a question you need to answer for yourself in order for our Definition 1 to be sufficient to cover your previous idea of an "even number", and this will be left to you.

From now on, let E represent the set of even numbers, i e , $E = \{0,2,4,6,\dots\}$.

Consider the pattern of products of 2 again.

$$0 = 2 \times 0$$
$$2 = 2 \times 1$$
$$4 = 2 \times 2$$
$$6 = 2 \times 3$$
$$\vdots$$

Can we generalize this pattern to get an expression which always represents an even number? We have touched on this in a previous thought exercise. Suppose we let n represent any whole number, i.e., $n \in W$. From the pattern above and our definition of even number, can we say "$2 \times n$ is an even number"? A statement in symbols equivalent to Definition 1 would be

If n is a whole number $(n \in W)$ then
$2 \times n$ is an even number $(2 \times n) \in E)$.

Before we turn our attention to the odd numbers, consider the following exercises which will check your understanding of the definition of even number.

Exercises:

1. Is each of the following numbers an even number--610, 324, 1024? How could you show whether each is or is not by use of Definition 1?

2. Again by use of Definition 1, can you show that $4 \times n$, where $n \in W$, is an even number? What about $6 \times m$ $(m \in W)$? $36 \times k$ $(k \in W)$?

3. Thought Exercise. Can you show that 25 is not an even number by use of Definition 1?

Odd Numbers

Think about the following statement. If we take an even number and add one (1), we get an odd number." If we look at some examples, $2 + 1 = 3$, $6 + 1 = 7$, $24 + 1 = 25$, it appears the statement is reasonable

Then suppose we consider the following pattern

$$0 + 1 = (2 \times 0) + 1 = 1$$
$$2 + 1 = (2 \times 1) + 1 = 3$$
$$4 + 1 = (2 \times 2) + 1 = 5$$
$$6 + 1 = (2 \times 3) + 1 = 7$$
$$\vdots$$

On the right we have each odd number and on the left each odd number written as an even number plus 1.

Do you notice a familiar term in each of these expressions? We see terms such as 2×0, 2×1, 2×2, 2×3, ... Remember we have agreed to let $2 \times n$ represent any even number. With this in mind, what does $(2 \times n) + 1$ represent? It obviously is an odd number.

So we can let $(2 \times n) + 1$ rppresent any number which occurs in the middle in our above listing of odd numbers. Then in general

If n is a whole number $(n \in W)$ then $(2 \times n) + 1$ is an odd number.

Thought Exercise: Using the expression $(2 \times n) + 1$ and substituting, one at a time, all whole numbers, will we obtain every odd number? (Remember

the conclusion of the other thought exercises.) Try to demonstrate this by trying some numbers and see if you can arrive at a conclusion.

In trying to answer this exercise, did you feel some uneasiness about lack of support for your answer ? If you did, this probably stems from the fact that the set asked about, the set of odd numbers, can never be completely written out. So by mere listing we could never prove anything about this set. A more precise definition of odd numbers is probably in order so we can think more intelligently about these numbers. In the same manner as Definition 1 of even number, we define an odd number as follows.

Definition 2. An odd number is any whole number which is one more than the product of a whole number and the number two.

This definition tells explicitly what an odd number is but if we are given a number it doesn't tell us how to determine whether it is odd or not. Suppose we consider even numbers again. 0, 2, 4, 6, ... If we divide each of these by 2 we get a remainder of 0, and 2 divides each "exactly". (Actually this is the same as our earlier definition using the inverse operation.) Thinking in this manner, what is the remainder when we divide each odd number by 2 ? Suppose we base a definition of odd number on this idea.

Definition 3 . An odd number is a whole number which leaves a remainder of 1 upon division by 2 .

This definition is actually stating the same thing as Definition 2 only in different words. Do you see a relationship between this definition and the expression (2 x n) + 1 for an odd number? What would we get if we divided (2 x n) + 1 by 2 ?

Now reconsider the previous "Thought Exercise". Can we now answer this question more positively ? Does every odd number leave a remainder of 1 on division by 2 ? Does (2 x n) + 1 leave a remainder of 1 on division by 2 ?

Sums and Products of Odd and Even Numbers

Suppose we pose some questions about sums and products.

1. Is the sum of two even numbers always even, always odd, or sometimes one and sometimes the other?

2. Is the product of two even numbers always even, always odd, or sometimes one and sometimes the other?

3. Is the sum of two odd numbers always even, always odd, or sometimes one and sometimes the other?

4. Is the product of two odd numbers always even, always odd, or sometimes one and sometimes the other?

Let us seek an answer to the first in the following manner.

1.) $0 + 0 = 0$ $\quad$ $2 + 2 = 4$ $\quad$ $4 + 4 = 8$

$0 + 2 = 2$ $\quad$ $2 + 4 = 6$ $\quad$ $4 + 6 = 10$

$0 + 4 = 4$ $\quad$ $2 + 6 = 8$ $\quad$ $4 + 8 = 12$

Apparently the answer is "Yes". The key word here is "apparently". We cannot be sure on the basis of the little information we have accumulated whether we are correct or not. We must be careful about basing conclusions in number theory (and in fact in all mathematics) on inductive reasoning (basing the conclusion on a few examples). Examples will be given later where this type of reasoning leads to false conclusions. To be very sure of our conclusion we must make a deductive argument as follows.

Remember that we can represent any even number as $2 \times$ (some whole number). In Question 1, we have the sum of two even numbers. Suppose we represent these by $2 \times n$ and $2 \times m$. (Why should we not use $2 \times n$ for both numbers?) Their sum is represented by

$$(2 \times n) + (2 \times m).$$

By the distributive law we have

$$2 \times (n + m).$$

Now n and m are whole numbers ($n \in W$ and $m \in W$) so what can we say about $n + m$? (This is using the closure property with respect to addition.)

Thus $2 \times (n + m)$ represents an even number since it is $2 \times$ (some whole number). And so $(2 \times n) + (2 \times m) = 2 \times (n + m)$ = an even number for any even numbers, $2 \times n$ and $2 \times m$. Thus we can answer the first question by 'the sum of two even numbers is always an even number'.

Let's consider the second question and try to provide it deductively. Again we have two even numbers, $2m$ and $2n$. Their product is $(2 \times m) \times (2 \times n)$. By the associative principle we have:

$$2 \times [\, m \times (2 \times n)].$$

Is $[\, m \times (2 \times n)]$ a whole number? Could you prove it? So we have $(2 \times m) \times (2 \times n) = 2 \times$ (some whole number) which is an even number. Thus the answer to Question 2 is, "The product of two even numbers is an even number."

Exercises: Using deductive methods, prove the following.

1. Answer (and prove) question numbers 3 and 4 above. (Remember an odd number is represented by $(2 \times n) + 1$ where $n \in W$).
2. The sum of an even number and an odd number is odd.
3. The product of an odd number and an even number is even.
4. The product of two odd numbers is odd.

The Sets of Even and Odd Numbers

Possibly now that we know something more about even and odd numbers, we should answer two questions that might have been brought to mind at

the beginning of this unit. These questions are : Are the two sets E and O, of even and odd numbers, respectively, disjoint, that is, do they have any common elements or not? Does their union make up the set of whole numbers?

The answer to the second of these can be seen very easily by relying on our intuitive ideas of even and odd numbers. The set of even numbers contains zero and every second number thereafter and the set of odd numbers contains 1 and every second number thereafter. Now think of combining these two sets. The odd numbers fill in the "gaps" of the set of even numbers and vice-versa. Remembering our general forms of even and odd numbers we may argue as follows. E contains $2n$, so it contains $2n + 2$, $2n + 4$, $2n + 6$, etc. (every second number thereafter). O contains $2n + 1$, $2n + 3$, $2n + 5$, etc. Do you see that the gap between $2n + 1$ and $2n + 3$ is filled by $2n + 2$? You should if we write $2n + 2$ in the following forms.

$$2n + 2 = (2n + 1) + 1$$

$$2n + 2 = (2n + 3) - 1$$

We see that $2n + 2$ is one more than $2n + 1$ and one less than $2n + 3$. So it is precisely the number which fills the "gap" between $2n + 1$ and $2n + 3$.

Do the following exercises to justify to yourself that we have indeed shown that $E \cup O = W$.

Exercises

1. Show that $2n + 4$ fills the "gap" between $2n + 3$ and $2n + 5$.

2. Show that $2n + 7$ fills the "gap" between $2n + 6$ and $2n + 8$.

3. What numbers fill the following gaps ? (Justify your answer.)

   Between $2n + 9$ and $2n + 11$.
   Between $2n + 16$ and $2n + 18$.
   Between $2n + 101$ and $2n + 103$.

The answer to the first question as to whether the sets are disjoint can be answered by a type of argument common in mathematics. In this type of argument, commonly called the indirect method of proof, we assume the opposite of what it is we wish to prove and see where it leads us. If this assumption leads to something false or contradictory, then we know that the assumption is not true. If the assumption is not true, then its opposite must be true and this opposite is precisely what we wished to prove at the beginning. We now proceed with the proof that the sets E and O are disjoint. We begin by assuming that they are not, (in symbols $E \cap O \neq \emptyset$). This means that these sets must have at least one common element. This is to say that there is some number that is both even and odd. This may appear an impossible occurrence, but remember that we wish to prove that this cannot hold and that the opposite statement holds, namely that there is no number which is both even and odd. Now we have assumed that we have a number which is an element of both the two sets E and O . That this number is in E means it is of the form $2 \times m$, for some $m \in W$. That this number is also in O means that it is of the form $(2 \times n) + 1$ for some $n \in W$. Thus $2 \times m$ and $(2 \times n) + 1$ represent the

same number and we can say

$$2 \times m = (2 \times n) + 1 .$$

We now need to apply some algebra and get

$$(2 \times m) - (2 \times n) = 1 .$$

or by using the distributive principle,

$$2 \times (m - n) = 1 .$$

Let's see what this says. If m is greater than n then m - n is a whole number. Then 2 x (m - n) is an even number. So the equation above says 1 is an even number, an absurd statement. If n = m, then m - n = 0 so 1 = 2 x 0 = 0, again an absurd statement. If n is greater than m, then m - n is a negative number so 2(m - n) is negative. Thus the equation says 1 is negative. So no matter what the relation between m and n, the assumption we started with, namely that our original number is in both E and O, cannot be. Thus there cannot be any number that is both even and odd.

Thus we see that

$$E \cup O = W$$

and $E \cap O = \emptyset$ (E and O are disjoint).

We have had to reach out of our set W of whole numbers into the negative integers in the above arguments, though at the beginning it was stated that our work would be all in W. The theory of numbers actually includes study of all the integers, positive, negative, and zero, so we

are not incorrect in using them; but we will not do so again.

2.2 Factors, Divisors and Multiples

In the last section we noted and made use of the fact that there is something which all even numbers have in common. Every even number can be written as $2 \times$ (some whole number) or $2 \times n$ where $n \in W$. We can indicate this by saying "2 is a factor of every even number." Equivalently, we can say, "2 is a divisor of every even number." We should try to get precise definitions of the terms "factor" and "divisor." To do this, we shall use the idea of "exact divisibility."

If we were to ask a child in the middle elementary school what it means to say one number divides another exactly be would probably know but might not be able to verbalize it. For example, we know

2 divides 4 exactly.
8 divides 32 exactly.
5 divides 75 exactly.

Also

2 does <u>not</u> divide 17 exactly.
8 does <u>not</u> divide 9 exactly.
5 does <u>not</u> divide 25 exactly.

But now we should ask: awhat criteria do we use to determine exact divisibility? Probably we divide and check the remainder. More precisely we could make a definition.

Definition 4. Given m and n are whole numbers, we say m divides n exactly if, when n is divided by m, the remainder is zero. (We should of course exclude m = 0 since we can never divide by zero.)

Another definition of dividing exactly which may be less workable at the moment but applicable later is possible. It is based on the idea that if m divides n exactly then there is a whole number which is the quotient. For example

2 divides 8 exactly because $8 = 2 \times 4$. (4 is the quotient)

5 divides 75 exactly because $75 = 5 \times 15$. (15 is the quotient)

In general, then, we have a definition following this pattern.

Definition 5. m divides n exactly if and only if there is a whole number k such that $n = m \times k$. (k is the quotient)

Now we can define factor and divisor.

Definition 6. If m and n are whole numbers, then m is a whole number factor (divisor) of n if and only if m divides n exactly.

We must be careful to specify the kind of factors of which we are speaking. In this unit we shall be concerned with only whole number factors. It should not be thought that this is the only type. For example, though 2 and 1 are factors of 2 so also are 1/2 and 4 since $1/2 \times 4 = 2$ and also are -3/8 and -16/3 since $-3/8 \times -16/3 = 2$ and there are

many more such factors. The distinction between these examples occurs when we note that the first two are whole number factors and the last two pairs of factors are rational (or fractional) factors. So it is necessary to specify the kind of factors we are speaking of in a particular case and this will be done throughout the unit.

A concept closely related to factors and divisors is that of multiples. Consider the even numbers.

$$0 = 2 \times 0$$
$$2 = 2 \times 1$$
$$4 = 2 \times 2$$
$$6 = 2 \times 3$$
$$8 = 2 \times 4$$
$$\vdots$$
$$= 2 \times n$$

Examining this pattern, we note that we can say, "2 is a factor of every even number." Equivalently, we can say "every even number is a multiple of 2." Notice that to get the multiples of 2 we can count by twos.

Suppose we investigate multiples of 3 based on our intuitive notions of multiple.

Thought Exercise

List the first 6 multiples of 3 in increasing order. Is this like "counting by threes"? Did you list zero? Is zero a multiple of 3?

Notice above that we said every even number is a multiple of 2 and

we consider zero as an even number. Following this reasoning, zero is a multiple of 2. Is it possible zero is also a multiple of 3? Perhaps we should have a more precise definition of multiple than our intuitive notion affords to enable us to answer such questions.

We have seen that in the case of the number 2 there is a close relationship between factor and multiple. Suppose we investigate for such a relationship for 3 and 4. The information is arranged in a table for convenience.

n	Numbers of which n is a factor	Reason
3	0	$0 = 3 \times 0$
	3	$0 = 3 \times 1$
	6	$6 = 3 \times 2$
	9	$9 = 3 \times 3$
	⋮	⋮
4	0	$0 = 4 \times 0$
	4	$4 = 4 \times 1$
	8	$8 = 4 \times 2$
	12	$12 = 4 \times 3$
	⋮	⋮

Consider the numbers in the second column. Are the first four numbers multiples of 3? Are the second four numbers multiples of 4? Then the relationship between factor and multiple appears to hold for numbers other than two. We utilize this in the following definition.

Definition 7. If m and n are whole numbers and m is a factor of n, then n is a multiple of m.

An Equivalent definition of multiple related to Definition 5 of factor is as follows.

Definition 8. If m and n are whole numbers, n is a multiple of m if and only if $n = k \times m$ for some whole number k.

Working the following exerciese should help clarify the above ideas and also point to later developments.

Exercises:

1. Make a list of the first 4 elements of the sets of multiples of each of 5, 6, and 7. Show each in the form of the third column of the table above. For example, $10 = 5 \times 2$, $18 = 6 \times 3$, etc. Remember that this shows that 5, 6, or 7 are factors of these numbers and that these numbers are multiples of 5, 6, or 7.

2. Do you remember the generalization that was made of the pattern which was seen for even numbers? We had

$0 = 2 \times 0$
$2 = 2 \times 1$
$4 = 2 \times 2$
$6 = 2 \times 3$
⋮

This was generalized to the fact that every even number can be written

as 2 x n. What pattern do you see for 3, 4, 5, 6, and 7? Can you write a general element of the sets of multiples of these numbers?

3. Of what whole number is 6 a multiple? Is there more than one? Show this. What numbers are factors of 6? Show this. Of what whole number is 12 a multiple? Is there more than one? Show this. What are the factors of 12?

4. Can you find a whole number which is a multiple of 2, 5, and 7? Show how you do it. What are the factors of this number? Make sure you have them all.

5. Is zero a factor of any whole number? A multiple? Is any whole number a factor of zero? A multiple?

6. Complete the following table.

n	factor of n	Reason
1	1	1xx 1 = 1
2	1, 2	1 x 2 = 2
3	1, 3	1 x 3 = 3
4	1, 4	1 x 4 = 4
	2	2 x 2 = 4
5	.	.
6	.	.
	.	.
7		
8		
9		
10		
0		

What whole number appears to be a factor of every other whole number? Can you generalize this for any whole number n ?

7. We can represent the set of multiples of a number many ways. For example consider the set of multiples of 3. We can write

$A = \{0, 3, 6, 9, 12 \ldots\}$.

We might also write the set of multiples of three by the following.

$B = \{3n \mid n \text{ is a whole number}\}$

(the | is read "such that"})

$C = \{3n \mid n \in W\}$

D = set of all whole number multiples of 3.

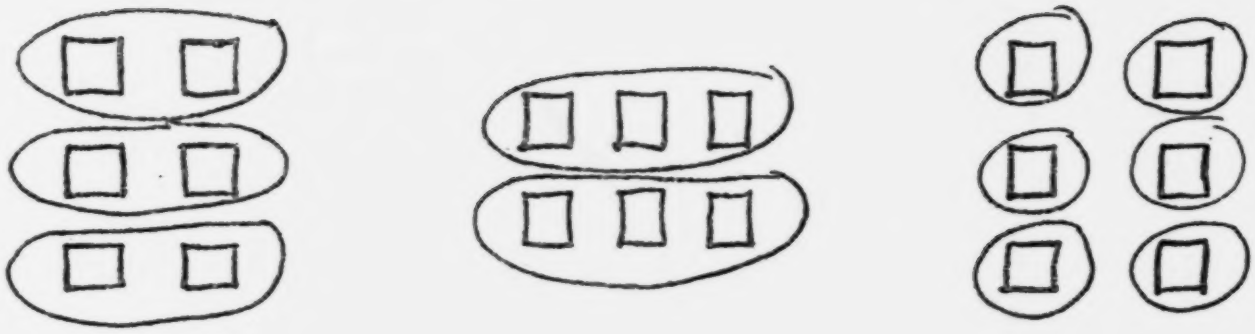
Verify to your own satisfaction that $A = B = C = D$; that is, that each of these sets contain exactly the same elements.

2.3 Prime and Composite Numbers

The concept of a prime number is one of the fundamental ideas of number theory. Some of the most beautiful and profound theorems and results in this field of mathematics are about prime numbers. Historically, the study of prime numbers has been very fruitful to the field of number theory. Tools have been developed in dealing with problems about primes which have been extremely valuable in other areas. The Greek Pythagoreans attributed magical powers to primes and much mysticism to their study.

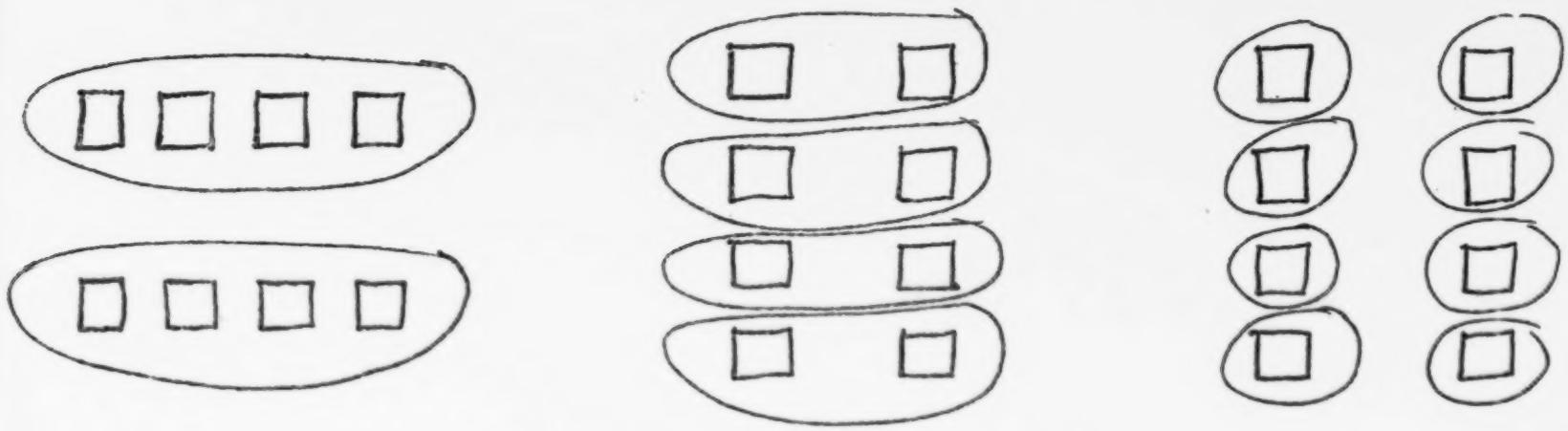
Today the mysteries and fascinations of prime numbers still intrigue and occupy mathematicians. Some problems dealing with primes which when stated are simple enough to be understood by elementary students and yet they have defied solution for many centuries.

Before we get into precise developments about primes it might be instructive to see how these concepts can be introduced to elementary school children. The ideas of what prime numbers are can be introduced concretely and easily to elementary school children by the use of sets of blocks. If they are given sets of blocks containing different numbers of blocks, say 4, 5, 6, 7, and 8, and they are asked to take each set and divide up the blocks in that set into equal piles, they can be lead to a basic understanding of prime numbers as well as numbers which are not prime. This can be accomplished as follows: Suppose a child is given a set of 4 blocks. If he is asked to divide this set into a number of piles (sets) each containing the same number of blocks, he might divide it into 2 piles (sets) of 2 blocks each. If another child is given the same task, he may divide the set into 4 sets each with one block. In a classroom there will surely be students coming up with each solution. This is the activity for which we are looking. (A precocious student might suggest that one pile of 4 is an answer, though by the way the task is described this would not be anticipated.) Suppose we now give the students another set of blocks containing 6 blocks and again ask them to divide it into equal sets. Some may get three sets of 2 each, some two sets of 3 each and some six sets of 1 each.



Suppose we try a set of 8 blocks. Here we could get 2 sets of 4 each.

4 sets of 2 each and 8 sets of 1 each.

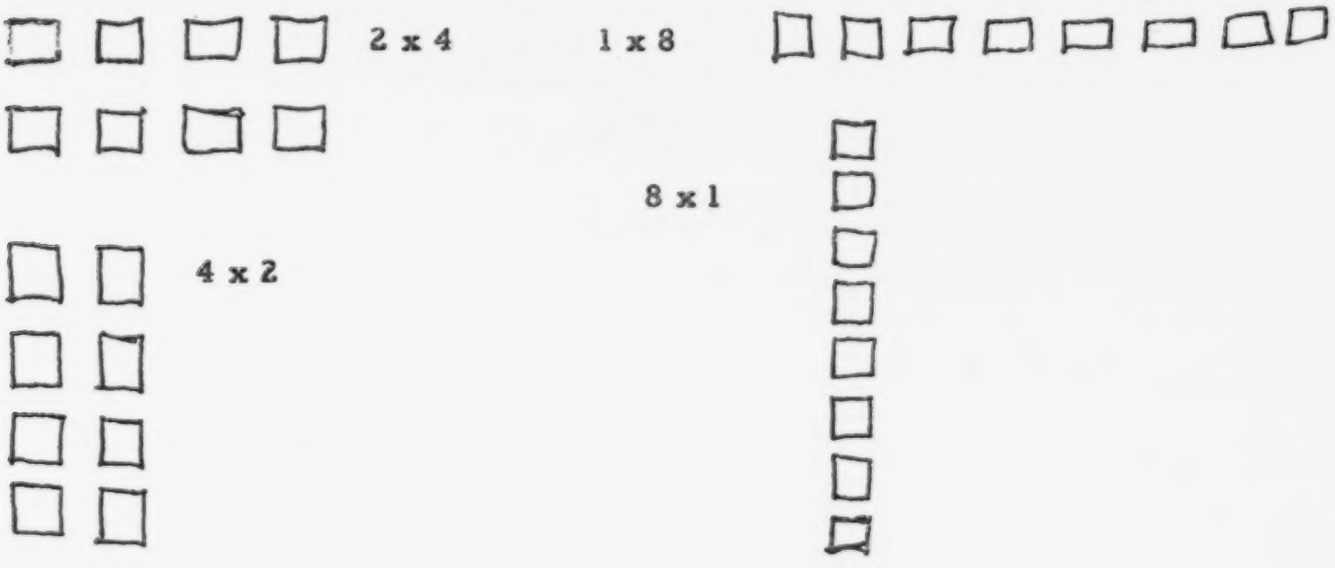


Such results might lead to a conjecture from the students that if we take any set of blocks there are many ways that we can divide it into equal sets of blocks. It is not necessary that they make such guesses though they should be asked questions about any patterns they might see. Now let them try a set of five blocks. After trial and error, they will come up with only one way of getting sets of an equal number of blocks. (They should be encouraged to try for more than one way since this is the crucual point of the experiment.) (Again some precocious student may suggest one pile of five blocks and say that there really is two ways

though when we get into prime numbers and factors we will see that these are the same concept.) Now the students should be allowed to try sets of blocks containing different numbers up to say 20. A table should be kept of each of the different arrangements for each number as they are found. After several numbers are tried then we can make some kind of definition of prime number in relation to the number of ways we can divide it into equal sets. It might take the form of "a number is prime if we can divide it up into equal sets in only one way (excluding the case of only one set of all the blocks)". After these ideas have been developed then one could go into a more precise definition of prime such as we will do in this section.

A variation of the above experimental approach to primes would be to ask the students to arrange the sets of blocks into a rectangular form of so many rows with an equal number in each row. This is still dividing the sets into a number of equal sets, but it may be more convincing and concrete to the students to see the equal sets arranged neatly in rows.

For example, for 8 we might have



You will notice that the one set of 8 blocks is sure to mentioned here as one row of 8 blocks so the definition of prime would have to allow two ways of arranging the blocks into equal rows. There is no inconsistency between these two approaches to primes but students might be confused if both methods were used. We now proceed with more precise mathematical developments about prime numbers. We shall begin our study of these based on what we have learned previously about factors, divisors and multiples.

Consider the table which you completed in Exercise 6 of the previous section. It appears that every whole number except one (1) has at least 2 distinct factors. Your answer to the question in this exercise should support this. For any whole number n,

$$n = n \times 1 .$$

Hence by our definition of factor, n and 1 are both factors of n. Notice also that there are some numbers which have more than two factors, for example 4, 6, 8, 9, 10. (Remember the arrangements of blocks?) These facts will lead to our definition of prime numbers.

Definition 9. A prime number is any whole number which has exactly two distinct factors.

Thus from our list in Exercise 6 above, the primes less than 11 are 2, 3, 5, and 7.

For a moment, we will consider the whole numbers which are not prime. These we break up into two sets: those which do not have two distinct number factors, and those with more than two whole number factors. Can you decide what numbers should be in the first set? It is obvious that I must be in the first set if you ask yourself what two whole numbers can be multiplied together to get 1. Only $1 \times 1 = 1$ so 1 does not have two distinct whole number factors. Are there any other numbers in the first set? Remember that we said that any whole number, n, other than one has at least two factors, 1 and n. So 1 is the only number in the first set. In what set shall we place zero? If you completed the table in Exercise 6, Section 2.2, correctly, you should see the answer. Consider the following.

$0 = 0 \times 0$

$0 = 1 \times 0$

$0 = 2 \times 0$

$0 = 3 \times 0$

$0 = 4 \times 0$

$\vdots$

$0 = n \times 0$ for any whole number .

Hence every whole number is a factor of zero. How many such factors are there? Could you ever count them all? We indicate this by saying there is an infinite number (this means more than we could count). So we might say that zero is in the last set but it will cause confusion to include it in this set so we generally put it in a set by itself. What we are saying then is that all whole numbers other than 0, 1, and the primes are in the second set. This set of numbers is called the set of composite

numbers and we will denote the set by C. A definition is in order.

Definition 10. A composite number is any whole number which has more than two distinct whole number factors. (This of course excludes an infinite number of whole number factors.)

Another way of characterizing these numbers is by:

Definition 11. A composite number is any whole number other than 0 and 1 which is not prime.

We now have another decomposition of the set of whole numbers into sets. If we denote the set of primes by P then we have

$$W = P \cup C \cup \{1\} \cup \{0\} .$$

Actually these sets are also disjoint because a whole number has either 1, 2, more than 2 factors, or an infinite number of factors and this places it in one and only one of the above sets. (Remember we said any whole number greater than 1 has at least 2 factors.) This is an application of a very important principle of mathematics called the Fundamental Theorem of Arithmetic. This will be discussed in Sec. 2.5.

A question which first arises in, "How can we tell whether a number is prime or composite?" The most obvious way to tell is to try to find factors of it. This becomes increasingly difficult, if not impossible, as numbers increase in size. For instance, it is relatively easy to

determine primes and composites less than 100. It is more difficult to determine those between 100 and 1000 and almost impossible without the use of a computor for very large numbers. An ingenious device for determining primes was invented by Eratosthenes (c. 230 B. C.). This device is called a sieve and is constructed as follows. We begin by writing down the whole numbers in order.

1 (2) (3) ~~4~~ (5) ~~6~~ (7) ~~8~~ ~~9~~ ~~10~~
(11) ~~12~~ (13) ~~14~~ ~~15~~ ~~16~~ (17) ~~18~~ (19) ~~20~~
~~21~~ ~~22~~ (23) ~~24~~ ~~25~~ ~~26~~ ~~27~~ ~~28~~ (29) ~~30~~
(31) ~~32~~ . . .

We note the first prime, 2, circle it and cross out every second number thereafter. (Remember the relationship between factors and counting by a certain number?) We note the first number not crossed out, 3, circle it and cross out every third number thereafter, (Some may have already been crossed out.) Five is the next number not crossed out, we circle it and cross out every fifth number thereafter. And so on. Note that every circled number is a prime and each time the next prime is the next number which has not been crossed out. Can you see why the circled numbers are all the primes? Each crossed out number has factors other than itself and 1 and each circled number does not. This sieve of course becomes very unwieldy for large numbers. Other ways of finding primes have been sought after. Men have searched for centuries to find formulas for determining prime numbers. The best that have

been obtained are formulas which approximate or indicate the approximate number of primes less than a certain number. What is meant by this is best explained by some examples. Consider 10. The primes less that 10 are 2, 3, 5, and 7. They are four in number. The primes less than 20 are 2, 3, 5, 7, 11, 13, 17, and 19, eight in number. Less than 30 we have those less than 20 plus 23 and 29, ten in number and so on. Less than 100 there are 25 prime numbers. As we take successively larger numbers there appears to be no connection between the size of the number and the number of primes less than it except the larger the number the more primes less than it. The formula mentioned above has been known since 1800 and gives a rough approximation to the number of primes less than any given number. We will not discuss this formula here as it involves ideas beyond the scope of our work. It suffices to know that such exists. No formula is known which gives exactly the number of primes less than a given number.

Another question which arises is "How many primes are there?" We seem to get an indication to the answer by looking at the number of primes in intervals between numbers. For example, between 1 and 100 there are 25 prime numbers including 2, 3, 5, 7, 11, 13, 17, 19 and so on. Between 100 and 200 there are 21 prime numbers and between 200 and 300 there are 16. The table below gives the number of primes in intervals of 100 numbers up to 1000. If we continued this table on out to 100,000 it would be very evident that the numbers in each 100 number interval is getting less.

Numbers from	1	100	200	300	400	500	600	700	800	900
to	100	200	300	400	500	600	700	800	900	1000
Number of Primes in the Interval	25	21	16	16	17	14	16	14	15	14

This "thinning out" might lead us to conjecture that there is a certain number of primes and that after a certain point we would find no more primes. As has been pointed out before, conjecture based on a few specific cases must be avoided in mathematics because this many times leads to false conclusions. This is a very good example of this fact, for it was proved by Euclid many centuries ago that the number of primes is infinite. This is to say no matter how many primes are found, there will always be more. The proof of this theorem is very simple and can be found in the appendix for any who wish to follow it through. Many large primes have been found and tables of primes have been constructed up to 10,000,000. In 1961, the largest verified prime was $2^{3217}-1$, a number which, when written in usual form, contains 969 digits.

The following exercise which may ask some of the many interesting questions which may be raised about the relationship between the sets E and O of even and odd numbers and the sets P and C of prime and composite numbers.

Thought Exercise: What are the relationships between the set E and the set P ? Are there any even numbers that are prime, i.e., what is

is

$E \cap P$? What about $E \cap C$, $O \cap C$, and $O \cap P$?

Don't be concerned if you can't answer exactly; just think about these and try to indicate what each set would be.

2.4 Factorization

We have seen that all whole numbers greater than 1 have two or more distinct factors and we have indicated this by showing these numbers as a product of two whole numbers. For example,

$2 = 2 \times 1$
$4 = 2 \times 2 = 4 \times 1$
$9 = 3 \times 3 = 9 \times 1$.

This process is very important in mathematics and has been given a name. Whenever we show a number as the product of two or more factors, we say we have "factored" the number. The process we call "factoring". The representation of a number as the product of two or more factors we call a "factorization" of the number. For example, if we "factor" 8, we get

$$8 = 2 \times 4 .$$

8 has been "factored" into 2 x 4 which is a "factorization" of 8. Notice we say "a" factorization of 8. Are there other factorizations of 8? Of course, $8 = 8 \times 1$. Also, $8 = 2 \times 2 \times 2$. So there are at least three factorizations of 8. Are there more? What shall we say about $8 = 1 \times 8$

and $8 = 4 \times 2$? It is conventional to call 8×1 and 1×8 the same factorization of 8. Also 4×2 and 2×4 are the same factorization of 8. (What principle of arithmetic is being applied here?) Shall we now say that 8 has only three factorizations: 4×2, 8×1 and $2 \times 2 \times 2$? What about $(1/2) \times (16)$, $(3/4) \times (32/3)$, and $(-2) \times (-4)$? Are these not factorizations of 8? We must be careful to specify what set of numbers we are using as factors or what set of numbers we are factoring over. We call this set the "domain of factorization". Thus if we ask, "What are the factorizations of 8 if the domain of factorization is the whole numbers?" then we can answer the above questions definitely. The only factorizations in this case are 4×2, 8×1, and $2 \times 2 \times 2$.

The following exercises will help acquaint you with these terms.

Exercises: (The domain of factorization is the set of whole numbers).

1. Find all factorizations of 16. What are the factors of 16?

2. Find all factorizations of 4. What are the factors of 4?

3. Do you see a relationship between the factorizations of 4 and 16? what is it?

4. Consider the following:

$$8 = 8 \times 1, \quad 8 = 8 \times 1 \times 1, \quad 8 = 8 \times 1 \times 1 \times 1 \times \ldots$$

Does this suggest some further modification or clarification of the term "factoring"?

2.5 Prime Factorization

Consider the following factorizations of composite numbers 1 to 20.

$4 = 2 \times 2 = 1 \times 4$

$6 = 2 \times 3 = 1 \times 6$

$8 = 2 \times 2 \times 2 = 2 \times 4 = 1 \times 8$

$9 = 3 \times 3 = 1 \times 9$

$10 = 2 \times 5 = 1 \times 10$

$12 = 2 \times 2 \times 3 = 2 \times 6 = 3 \times 4 = 1 \times 12$

$14 = 2 \times 7 = 1 \times 14$

$15 = 3 \times 5 = 1 \times 15$

$16 = 2 \times 2 \times 2 \times 2 = 2 \times 8 = 4 \times 4 = 1 \times 16$

$18 = 2 \times 3 \times 3 = 2 \times 9 = 3 \times 6 = 1 \times 18$

$20 = 2 \times 2 \times 5 = 2 \times 10 = 4 \times 5 = 1 \times 20$

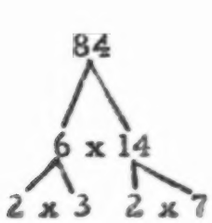
What do you notice about the factors in the first factorization in each case? Are these factors prime or composite? It appears that each composite number has a factorization in which all factors are prime. We wish to single this factorization out to be used in further developments so we call this the prime factorization of a number. Notice we can say the prime factorization because we have agreed to call factorizations such as 2×5 and 5×2, $2 \times 3 \times 3$ and $3 \times 2 \times 3$ the same factorizations of 10 and 18 respectively. Also in saying the prime factorization we are applying a very important theorem which is basic to many branches of mathematics. This is the

Fundamental Theorem of Arithmetic (also called the Unique Factorization Theorem): Any composite whole number other than 0 and 1 can be factored into primes in one and only one way, except possibly for the order in which the factors occur.

This means that if we factor a whole number n by a x b x c x d where these are primes, then any other factorization of n into primes must have the same prime factors, a, b, c, and d, possibly in some other order. It may appear that this theorem is a very simple result and is obvious. It is also very important as there are some mathematical systems in which unique prime factorization does not hold, though we do not encounter any in the elementary school program. The proof of this theorem is relatively simple but it would not suit our purposes to present it here. For the interested student a proof may be found in any theory of numbers text.

Students often encounter difficulty in finding the prime factorization of numbers. A device helpful in this is called a factor tree and is illustrated by steps as follows. Suppose we wish to factor 84. We might notice that 6 will divide 84. So we factor 84 using 6 as one factor.

Then we find factors of 6 and 14,

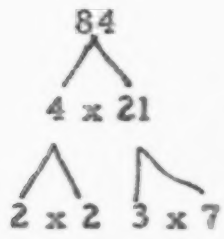


Notice that we have continued until we have all prime numbers as factors.

So $84 = 2 \times 3 \times 2 \times 7$.

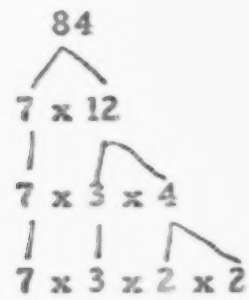
We could also have proceeded by noticing first that 4 is a factor.

The factor tree thus obtained is



So $84 = 2 \times 2 \times 3 \times 7$

Also



So $84 = 7 \times 3 \times 2 \times 2$

Notice that no matter which tree we use, we get the same factorization into primes except for order. This is an evidence of the Fundamental Theorem of Arithmetic.

The factor tree is a very concrete and graphic method for illustrating to students the meaning of factorization. Sometimes this method of factoring is not easy to initiate, especially with large numbers. There is a more systematic way of factoring a number. This is illustrated by the following example.

Again let us factor 84. We begin with the smallest prime, 2, and see whether or not it is a factor of 84. We see that it is and write

$$84 = 2 \times 42.$$

We now try to factor 42 and we again try the smallest prime, 2. It is a factor so we write

$$84 = 2 \times 2 \times 21.$$

If we try 2 again we see it is not a factor of 21 so we try the next prime, 3, and we get

$$84 = 2 \times 2 \times 3 \times 7.$$

At this stage all the factors are prime so we are done. (Note again the illustration of the Fundamental Theorem.)

As another illustration, suppose we factor 630. Again we try consecutive primes 2, 3, 5, 7, etc. Remember each may be a factor more than once.

$$\begin{aligned} 630 &= 2 \times 315 \\ &= 2 \times 3 \times 105 \\ &= 2 \times 3 \times 3 \times 35 \\ &= 2 \times 3 \times 3 \times 5 \times 7. \end{aligned}$$

Actually, with this method we are trying each consecutive prime as a divisor of the composite number. For this reason we may call it the

consecutive prime method. We can carry out this process then as a continued division by consecutive primes.

$$
\begin{array}{r}
7 \\
5\overline{)35} \\
3\overline{)105} \\
3\overline{)315} \\
2\overline{)630}
\end{array}
$$

So $630 = 2 \times 3 \times 3 \times 5 \times 7$.

In illustrating this to students, one should probably not go from the original discussion to the final algorithm form too quickly. The student should understand the reasons why the method works and the reason for each step before he makes a mechanical operation (the repeated division algorithm) of it.

We will apply prime factorization throughout the remainder of this booklet and the reader should be familiar with it before proceeding. The following exercises will help in this respect.

Exercises: (The domain of factoring is the set of whole numbers.)

1. Find the prime factorization of the following numbers by two methods.

   a. 32
   b. 18
   c. 38
   d. 16
   e. 36
   f. 272
   g. 45
   h. 25
   i. 700
   j. 75

Exponential notation. We sometimes use a shorthand notation in factoring into primes. To indicate the number of times any prime is a factor in the factorization, we use an exponent. Consider the following.

$45 = 3 \times 3 \times 5$

This can be written as

$45 = 3^2 \times 5$

The 2 is called an exponent and indicates the number of factors of 3 in the factorization. The 3 which is the factor is called the base .

Note: You must be precise in speaking of this. A common error by students is to say the exponent indicates the number of times you multiply 3 by itself. This is very confusing and incorrect as the following example shows.

Does $3^5 = 3 \times 3 \times 3 \times 3 \times 3$

or $3^5 = 3 \times 3 \times 3 \times 3 \times 3$?

In the first case, we have multiplied by 3 by itself five times and in the second case 3 is a factor five times. So we must be precise in our terminology and say the exponent indicates the number of times the base is used as a factor. If a number is a factor just once, then we use 1 as the exponent. The writing of the 1 in this case is optional and the base can be written without any exponent. Thus $3^2 \times 5^1 \times 7^1 = 3^2 \times 5 \times 7$.

2. Write each of the prime factorizations in Exercise 1 using the exponential notation.

3. What relationship do you see between the prime factorizations of 32 and 16? 18 and 36? 25 and 75?

4. Using each of the two methods given in this section find the prime factorization of 256. Of 425. Of 10,422, (indicate these in exponential notation).

Unit 3. GREATEST COMMON DIVISOR AND LEAST COMMON MULTIPLE

3.1 Introduction

We are all familiar with the processes involved in the following exercises.

$$\frac{3}{8} + \frac{5}{12} = \frac{9}{24} + \frac{10}{24} = \frac{19}{24}$$

$$\frac{15}{75} = \frac{1}{5}$$

We are also familiar with the difficulties encountered by many students in developing and retaining skill in carrying out these processes. The modern approach to teaching mathematics emphasizes understanding to show how the theory of numbers contributes to this understanding in the

above processes. First, we develop further some ideas which have previously been touched on and which will be useful in the following sections. These ideas deal with the divisibility of a whole number by a counting number.

3.2 The "Divides" Relation

In Definition 2, Section 2.2, we defined a relation involving division which can exist between two whole numbers. We repeat the difinition here for easy reference.

If $m \in W$ and $n \in W (m \neq 0)$, then m divides n exactly if and only if there is a whole number k such that $n = m \times k$.

We will drop the word "exactly" as it is mathematically correct to say "m divides n". Hereafter we will use a symbol to represent "divides". This is the vertical mark $|$. Thus we will replace m "divides" n by $m \mid n$. Our definition is then

If $m \in W (m \neq 0)$ and $n \in W$, then $m \mid n$ if and only if there is $k \in W$ such that $n = m \times k$

The properties of this relation are interesting and are easily understood when presented in a developmental manner. Suppose we first do some investigation with particular numbers.

The subscript, a device used continually in mathematics writing, is used here. What the subscript means is that the same letter with different subscripts represents different numbers. For example, k_1

and k_2 are used to represent values of k in the definition above which arise in different situations. k_1 might arise in the first example and k_2 in the second. The subscript is merely a tag to help us remember that k_1 and k_2 are values of k which have come about in two different instances. We are not saying by the use of the subscript that these two values of k could not be the same, i.e., we are not saying $k_1 \neq k_2$. These can be equal. We are merely distinguishing between them for the discussion.

We know that $4 \mid 8$ since $8 = 4 \times 2, (k_1 = 2)$.

Also $8 \mid 24$ since $24 = 8 \times 3, (k_2 = 3)$.

Does the relationship hold between 24 and 4 ? Yes, because

$$24 = 4 \times 6, (k_3 = 6).$$

Do you notice anything about the difference values of k ? Suppose we try another set of numbers. We know that

$7 \mid 28$ since $28 = 7 \times 4, (k_1 = 4)$

and $28 \mid 280$ since $280 = 28 \times 10, (k_2 = 10)$.

Now, does $7 \mid 280$? Yes, because

$$280 = 7 \times 40, (k_3 = 40)$$

Notice the k's again. Suppose we generalize this into a statement (called

a conjecture) and see if we can prove it.

If a, b, and c are whole numbers and $a|b$ and $b|c$, then $a|c$.

Let's analyze the above examples to see if we can discover a pattern which will lead to a proof of this statement for all whole numbers.

We say $7|28$ since $28 = 7 \times 4$. Analogously, can we say

$a|b$ means $b = a \times k_1$, where k_1 is some whole number?

Remember $a|b$ means there is a whole number k such that $b = a \times k$. We use k_1 to eliminate confusion with other values of k which follow.

Now we also said $28|280$ since $280 = 28 \times 10$. Let's carry the abalogy out with b and c.

$b|c$ means $c = b \times k_2$.

Now we have the two equations

1. $b = a \times k_1$ and 2. $c = b \times k_2$.

corresponding to 3. $28 = 7 \times 4$ and 4. $280 = 28 \times 10$.

Hopefully we can get

$c = a \times k$, where k is some whole number

because from this we get $a|c$. Notice that Equation 4 may be written $280 = (7 \times 4) \times 10$. Similarly, may we write

$c = (a \times k_1) \times k_2$?

Essentially, we are substituting $a \times k_1$ in place of b in equation 2 above which we may do since they name the same number. Now by the associative principle, we have

$$c = a \times (k_1 \times k_2)$$

and by closure,

$k_1 \times k_2$ is a whole number, let's call it k, that is let $k = k_1 \times k_2$.

Thus we have

$$c = a \times k$$

where k is a whole number. ($k = k_1 \times k_2$). Do you see the analogy to the above examples for the k's?) Thus $a \mid c$, so our conjecture has proved to be correct. (We may now call this a theorem.)

Let's try our hand at formulating another conjecture and proving it by considering numerical examples. Consider 4, 8, and 20.

We know

$4 \mid 8$ since $8 = 4 \times 2$, ($k_1 = 2$) and

$4 \mid 20$ since $20 = 4 \times 5$, ($k_2 = 5$).

Suppose we see if 4 divides the sum of these two numbers, that is, does $4 \mid (20 + 8)$.

$20 \div 8 = 28$

and yes $28 = 4 \times 7$.

Is this mere chance? Or is there a reason? Let's see if we can discover a reason for this.

Now $28 = 20 \div 8 = (4 \times 2) \div (4 \times 5)$

What principle of arithmetic can we use to show that 4 divides the right side of this equation, that is, to show that

$(4 \times 2) \div (4 \times 5) = 4 \times k$

for some whole number k ?

The distributive principle tells us that

$(4 \times 2) \div (4 \times 5) = 4 \times (2 \div 5)$.

Hence we have

$28 = 20 \div 8 = (4 \times 2) \div (4 \times 5) = 4 \times (2 \div 5) = 4 \times 7$, or $28 = 4 \times k_3$

where $k_3 = 7$.

Thus $4 \mid 28$

(Note the relationship between the values of k: 2, 5 and 7.)

Consider another example, 3, 27 and 18.

$3 \mid 27$ since $27 = 3 \times 9$

$3 \mid 18$ since $18 = 3 \times 6$

Does 3 divide the sum of 18 and 27? $18 + 27 = 45$, and yes it does. Let's see why.

$$45 = 27 + 18 = (3 \times 9) + (3 \times 6) = 3 \times (9 + 6) = 3 \times 15.$$

Hence $3 \mid 45$ since $45 = 3 \times 15$.

Let's formulate a conjecture based on what we have observed above. We have had in each case one number which divides each of two other numbers. We have seen then that the first number divides the sum of the other two. Let's write this in symbols.

If $a \mid b$ and $a \mid c$ then $a \mid (b + c)$.

Following the pattern observed above, let's attempt a general proof.

We have $a \mid b$ and $a \mid c$.

From this, we have by one definition of what divides means,

$$b = a \times k_1 \quad \text{and} \quad c = a \times k_2 .$$

Now $$b + c = (a \times k_1) + (a \times k_2) .$$

Using the distributive principle

$$b + c = (a \times k_1) + (a \times k_2) = a \times (k_1 + k_2) .$$

Thus $b + c = a \times (k_1 + k_2) = a \times k$, where $k \in W$ and $k = k_1 + k_2$.

Hence $a|(b + c)$ and the conjecture is proved.

These examples illustrate the value of number theory as a vehicle for teaching ideas of proof to children. The numbers dealt with are the simplest ones and no complicated ideas of logic are involved. Also, the proofs themselves are very easily arrived at by searching for patterns. The following exercises will give you some practice in constructing such proofs.

Exercises: Following this procedure of building patterns from specific numbers prove the following conjectures.

1. If $a|b$ then $a|(b \times c)$ where c is any whole number.
2. If $a|(b + c)$ and $a|b$, then $a|c$.
3. If $a|b$ and $a|c$, then $a|(b - c)$. (To keep in the set of whole numbers, assume that b is greater than c.)

Division Algorithm

All students of arithmetic become acquainted with a much more general idea of division than the above presented one. There are relatively few "divisions" which "come out exactly" or which leave zero remainders (such as the above discussion centers on). Most divisions as experience shows, leave a remainder. We shall now consider a general principle of division which includes, as a special case, "exact division". Without this general principle, division, or the process we use to carry it out, would be impossible. To lead up to this general principle, let us again consider specific numbers.

We know that given two numbers, say 8 and 28, we can carry out a process called division on these numbers, e.g., 28 ÷ 8. We are also familiar with the method used. We first seek a multiple of 8 which is less than 28.

$$
\begin{array}{r}
3 \\
8\overline{)\,28} \\
\underline{24} = 3 \times 8 \\
4 = 28 - (3 \times 8)
\end{array}
$$

Note that this says that $28 = (3 \times 8) + 4$. Do you see a similarity to the way we "check" division problems? The answer we obtain has been called by many names, the most prevalent today being "quotient". If the division is not exact, then we have a number "left over" which we call the "remainder". Probably everyone would agree that, no matter what two numbers had been chosen, such statements as those above could be made and that the operations could have been carried out. It would be a sad state of affairs if this were not the case. Very few people, though, know the justification for these statements and this operation. This justification is given by the following general principle, now stated.

The Division Algorithm: For any given whole numbers a and b, with $b \neq 0$, there are unique whole numbers q and r such that $a = (q \times b) + r$ and r is less than b.

Some illustrations of this might make the meaning of it clear. Consider 42 and 9. We can say

$42 = (4 \times 9) + 6$

also, $9 = (0 \times 42) + 9$

so we need not worry about which number we write as a and which as b to write it in the form of $a = (q \times b) + r$. Consider two more numbers, say 20 and 225. We can write

$225 = (11 \times 20) + 5$

or $25 = (0 \times 225) + 25$

Now let's analyze the statement of the Division Algorithm referring to a specific example given previously. We have two whole numbers a and b, $b \neq 0$ (Why?), just as we had 42 and 9 before. The Algorithm says that there are then two whole numbers, q and r (4 and 6) which are always the same for the given a ahd b , such that q is the quotient and r is the remainder when we divide a by b. Thus the Algorithm justifies the whole process of division and we are assured that, given any two whole numbers, we can divide one by the other (except by zero) and get an answer which will be the same for those two numbers every time we do this division.

The statement was made previously that the general Division Algorithm contained "exact division" as a special case. This is illustrated by letting $r = 0$. Then we have that

$a = (q \times b) + 0 = q \times b$

So $b|a$.

There is a small point of confusion which may develop here and should be pointed out. Sometimes the process which we carry out in division is called the "division algorithm". This means that we have a certain sequence of operations, closely related to the above Division Algorithm, which we memorize and carry out on numbers and which lead to a desired result. In this unit the word "division" will refer to these operations and the words Division Algorithm to the general mathematical principle.

Exercises: Analyze each of the following divisions to show the application of the Division Algorithm. (Write as $a = bq + r$ for some whole numbers a, b, q and r as in the Division Algorithm.)

1. $98 \div 12 = 6$ r 2

2. $125 \div 25 = 5$

3. $49 \div 5 = 9$ r 4

3.3 Least Common Multiple

We have previously defined multiple. However, for reference, we restate this definition:

> m is a whole number multiple of n if there is a whole number k such that $m = k \times n$.

Let us consider some whole number multiples of two numbers, say 8 and 12. Let E be a set of multiples of 8 and T be a set of multiples of 12.

$E = \{8, 16, 24, 32, 40, 48\}$

$T = \{12, 24, 36, 48\}$.

Notice E and T are not the sets of all multiples of 8 and 12 respectively. Also notice we do not list zero though it is a multiple of every number. In the situation we have now, zero as a multiple is of no value to us and would only cause confusion. What do you notice about these two sets? One thing we can readily note is that they are not disjoint, i. e.,

$E \cap T \neq \emptyset$, and in fact,

$E \cap T = \{24, 48\}$.

This means 24 and 48 are multiples of both 8 and 12 and to indicate this we say 24 and 48 are common multiples of 8 and 12. Are these the only common multiples of 8 and 12? Try to think of some more. In doing this, you should see a pattern related to some earlier work. Are all the common multiples of 8 and 12 a multiple of some number besides 8 and 12? Let's write a few to see.

24, 48, 72, 96, 120, ...

We can rewrite these as

1 x 24, 2 x 24, 3 x 24, 4 x 24, 5 x 24, ...

Can we generalize to say n x 24 is a common multiple of 8 and 12? It appears so. Later we will see that this is the case. It appears that 24 is a special common multiple of 8 and 12. It is the smallest one and it is a factor of all the other common multiples of 8 and 12. Such a number we distinguish by calling it the least common multiple. We record a definition for easy reference.

Definition 12 The whole number k which is a multiple of two different whole numbers m and n and is the smallest such number is called the least common multiple of m and n. Note that this number is the least number in the intersection of the sets of multiples of m and n.

What else can we say about the least common multiple, hereafter abbreviated l. c. m., of two numbers? We saw above that 24 is the l. c. m. of 8 and 12 and that every common multiple of 8 and 12 is a multiple of 24. Remembering the relationship between multiple and factor, can't we also say that 24 is a factor of every common multiple of 8 and 12? Do you suppose this relation holds in general? Let's consider another example.

The set of multiples of 5 is

$$\{ 5, 10, 15, 20, 25, \ldots \}$$

The set of multiples of 2 is

$$\{ 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, \ldots \}$$

The set of common multiples of 5 and 2 is

$$\{ 10, 20, 30, \ldots \}$$

So 10 is the l. c. m. of 5 and 2. It is also a factor of each multiple of 5 and 2.

It appears then that the l. c. m. of two numbers is a factor of every common multiple of these two numbers.

Thought exercise: Prove the conjecture that if x is the l. c. m. of a and b then x is a factor of every common multiple of a and b.

Now the question arises as to how to find the l. c. m. of two given numbers without listing sets of multiples of each and finding the intersection of these sets. Suppose we look at the sets of multiples of 8 and 12 (leaving out zero).

$E = \{ 8, 16, 24, 32, 40, 48, \ldots \}$

$T = \{ 12, 24, 36, 48, \ldots \}$

Remembering earlier developments we can generalize elements of each of these sets to

an element of E is of the form $8 \times n$ for $n \in W$ and

an element of T is of the form $12 \times m$ for $m \in W$.

Let's rewrite these in factored form as $8 \times n = 2 \times 2 \times 2 \times n$ and $12 \times m = 2 \times 2 \times 3 \times m$. Now by examination we can see that any multiple of 8 must have 2 as a factor 3 times. Also we can see that any multiple of 12 must have 2 as a factor twice and 3 as a factor once. We are really describing the elements of the two sets E and T with these statements. We are saying that each element of E must have 2 as a factor 3 times and that each element of T must have 2 as a factor twice and 3 as a factor once. How then can we describe the common multiples of 8 and 12, that is, the elements of the intersection of the

sets E and T ? How many factors of 2 must there be in a common multiple? Are 3 factors of 2 sufficient? How many factors of 3 must there be? Is one sufficient? Suppose we consider numbers with three factors of 2 and one factor of 3, that is, numbers of the form

$$2 \times 2 \times 2 \times 3 \times k \quad \text{for} \quad k \in W.$$

Notice that this is

$$2^3 \times 3 \times k = 24 \times k .$$

This is a multiple of 8 since $24 \times k = 8 \times (3 \times k)$ and it is also a multiple of 12 since $24 \times k = 12 \times (2 \times k)$. Hence we have a general representation for common multiples of 8 and 12, that is,

$$E \cap T = \{ x \mid x = 24 \times k \quad \text{for} \quad k \in W \} .$$

We find the l. c. m. by letting $k = 1$ since $24 \times k$ represents all common multiples and the smallest of these is when $k = 1$. (Remember we are not considering zero.)

The method which we used above of finding the l. c. m. of two numbers can be summarized as follows.

Suppose we wish to find the l. c. m. of 24 and 28.

1. Write the prime factorization of each number. $24 = 2^3 \times 3$ and $28 = 2^2 \times 7$.

2. For a multiple of 24 we need three factors of 2 and one of 3.

For a multiple of 28 we need two factors of 2 and one of 7.

3. Decide how many times each prime must be a factor of the number which is to be a common multiple.

For a common multiple we must have

2 as a factor at least 3 times because 2^3 is a factor of 24

3 as a factor at least 1 time because 3^1 is a factor of 24

7 as a factor at least 1 time because 7^1 is a factor of 28

So all common multiples are of the form

$2^3 \times 3 \times 7 \times k$ for $k \in W$.

To demonstrate that every number of this form is a common multiple of each of 24 and 28 we can write $2^3 \times 3 \times 7 \times k = (2^3 \times 3) \times 7 \times k = (24) \times (7 \times k)$ and $2^3 \times 3 \times 7 \times k = (2^2 \times 7) \times 2 \times 3 \times k = (28) \times (3 \times k)$. So the number is a common multiple of 24 and 28.

To find the l. c. m. we let $k = 1$.

$$2^3 \times 3 \times 7 \times 1 = 2^3 \times 3 \times 7 = 168.$$

This method can be abbreviated but this should not be done until students fully understand the reasons behind it.

Exercises:

1. Find the l. c. m. of 9 and 15, of 40 and 20, of 60 and 225.

2. Find the l. c m. of 2, 5, and 6. This is merely an extension of the method given above

3. In finding the l. c. m. of $8 = 2^3$ and $12 = 2^2 \times 3$, we took 2 as a factor only 3 times and 3 as a factor once. Can you explain why we do this instead of 2 as a factor 5 times and 3 as a factor once?

Least Common Denominator

The main application of the l. c. m. in arithmetic is in finding the least common denominator, l. c. d., of two rational numbers for purposes of combining them. For example, consider the problem with which we started this unit: $\frac{3}{8} + \frac{5}{12}$. We know that before we can combine these rational numbers we must change their representations to equivalent fractions which have the same denominators, that is, we must find a common denominator and convert those fractions into fractions with that denominator. For convenience we look for the least common denominator which happens to be the l. c. m. of the denominators of the two rational numbers we swish to combine. We have found that the l. c. m. of 8 and 12 is 24. This is why we convert as follows:

$$\frac{3}{8} = \frac{9}{24} \quad \text{and} \quad \frac{5}{12} = \frac{10}{24}$$

$$\text{So } \frac{3}{8} + \frac{5}{12} = \frac{9}{24} + \frac{10}{24} = \frac{19}{24}$$

If we would use another common multiple of 8 and 12, say 48, we would have

$\frac{3}{8} = \frac{18}{48}$ and $\frac{5}{12} = \frac{20}{48}$.

So $\frac{3}{8} + \frac{5}{12} = \frac{18}{48} + \frac{20}{48} = \frac{38}{48}$, an answer not in lowest terms.

This process is not wrong, and if a child can understand and use it, even if he has trouble with lthe l. c. d. process, he has a useful skill. However, the result $\frac{38}{48}$ is not in simplest form. It would probably require another step to this problem which would not be needed in the case where we use the l. c. m. instead of just some common multiple. Speaking of a fraction in lowest terms teads us into the next topic, the greatest common divisor of two numbers, which is closely related to the l. c. m.

3.4 Greatest Common Divisor

In the second of the problems which introduced this unit we reduce the fraction $\frac{15}{75}$ to lowest terms $\frac{1}{5}$. In doing so, we divided out the greatest common divisor (or factor), g. c. d., of 15 and 75. We shall be concerned here in showing exactly what the g. c. d. of two numbers is and three methods of determining it.

First, let's consider what the g. c. d. of two numbers, say 72 and 90 might be. First of all, it will be a divisor so let's list the sets of divisors of 72 and 90, call these S and N respectively.

$S = \{ 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72 \}$

$N = \{ 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90 \}$

Note that these sets are finite, that is, they have a finite number , 12 in

each case, of elements. How is this different from the set of multiples of a number? Is there only a finite number of multiples of any number? The set of multiples of a number is all products of that number and whole numbers, so it cannot be finite. Now back to divisors. The divisor of a number must be smaller than the number. Thus the set of divisors necessarily contains only a finite number of elements.

Second, the g. c. d. of 72 and 90 will be a common divisor of 72 and 90. Let's take the intersection of the sets S and N to find common divisors of 72 and 90.

$$S \cap N = \{1, 2, 3, 6, 9, 18\}.$$

How does the intersection of the sets of multiples of two numbers compare with this? For one thing, this set, $S \cap N$, is finite and the set of common multiples of two numbers is infinite. For example, the set of multiples of 4 is (excluding 0)

$$\{4, 8, 12, 16, 20, 24, \ldots\}$$

and of 5 is

$$\{5, 10, 15, 20, 25, 30, 35, \ldots\}.$$

The set of common multiples of 4 and 5 is

$$\{20, 40, 60, \ldots\}.$$

Remember that this happends to be the set of multiples of 20 so it is an

infinite set. Note that both sets have a smallest element: 1 for the set of common divisors and the l. c. m. for the set of multiples.

Third, the g. c. d. will be the greatest number in the set of common divisors of 72 and 90. Thus, the g. c. d. of 72 and 90 is 18.

Now, using these ideas we define, g. c. d.

Definition 13. The largest number which divides two whole numbers is called their greatest common divisor.
It is standard to symbolize the g. c. d. of two numbers a and b by (a, b).

Look at the set of common divisors, $S \cap N$, of 72 and 90 in relation to 18. Each of the common divisors divide 18. Might this be true in general? Suppose we investigate the factorizations of these numbers to see if we can see why each common factor divides the g. c. d.

$72 = 2^3 \times 3^2$ and

$90 = 2 \times 3^2 \times 5$

We have seen that $18 = 2 \times 3^2$ is the g. c. d. of 72 and 90. Suppose we regroup the factorizations of 72 and 90 so that the factor of $18 = 2 \times 3^2$ is obvious.

$72 = (2 \times 3^2) \times 2^2$

$90 = (2 \times 3^2) \times 5$

Note that there are no common factors of 72 and 90 other than (2×3^2). This is as it should be since $2 \times 3^2 = 18$ is the g. c. d. This also means that any common factor of 72 and 90 must be some number with no more prime factors than one factor of 2 and two factors of 3. Hence, any common factor of 72 and 90 must have the same factors as 18 and in equal or fewer number. Thus every common factor must divide 18. This discussion should suggest another method of finding the g. c. d. of two numbers besides finding the intersection of two sets of divisors. Consider 120 and 108.

Factor these numbers into prime factors.

$$120 = 2^3 \times 3 \times 5$$

$$108 = 2^2 \times 3^3$$

We see that for a number to be a common factor of 120 and 108, the greatest number of factors of 2 it can have is two and the greatest number of factors of 3 it can have is one. It can have no factor of 5 because then it would not be a factor of 108.

Hence, the greatest common factor will be

$$2^2 \times 3 = 12$$

We can see this clearly if we regroup the factors of 108 and 120.

$$108 = (2^2 \times 3) \times 3^2$$

$$120 = (2^2 \times 3) \times 2 \times 5$$

There are no common factors other than the two 2's and the one 3.

Let's check our generalization about all common factors dividing the g. c. d.

The set of factors of 108 is

$$A = \{1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54, 108\}$$

The set of factors of 120 is

$$B = \{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 24, 30, 40, 60, 120\}$$

The set of common factors is

$$A \cap B = \{1, 2, 3, 4, 6, 12\}.$$

Each common factor divides the g. c. d. This generalization always holds and can be used as an alternate definition of g. c. d.

Definition 14. The g. c. d. of two whole numbers is a whole number g, which has the following two properties:

1. It divides both numbers.
2. Any common factor of the two numbers divides g.

Exercises: Find the g. c. d. of the following pairs of numbers by two methods.

1. (8, 16)
2. (25, 45)
3. (108, 48)
4. (54, 36)

Euclidean Algorithm

An alternate method of finding the g. c. d. of two numbers which closely resembles the process of division and is in fact based on the Division Algorithm has been developed. This method is called the Euclidean Algorithm. A specific example follows.

Suppose we find the g. c. d. of 356 and 96.

1. We first divide 96 into 356.

$$96 \overline{)\,356}\quad \begin{array}{r} 3 \\ 288 \\ \hline 68 \end{array}$$

2. Next, we divide the remainder of this division into 96.

$$68 \overline{)\,96}\quad \begin{array}{r} 1 \\ 68 \\ \hline 28 \end{array}$$

3. Again, we divide the remainder into the divisor.

$$28 \overline{)\,68}\quad \begin{array}{r} 2 \\ 56 \\ \hline 12 \end{array}$$

4. Again

$$12 \overline{)\,28}\quad \begin{array}{r} 2 \\ 24 \\ \hline 4 \end{array}$$

5. Again

$$4 \overline{)\,12}\quad \begin{array}{r} 3 \\ 12 \\ \hline 0 \end{array}$$

This process terminates when we get a remainder of 0. The g. c. d. of the two numbers is the previous remainder, in this case 4.

Let's check by a previous method to see if 4 is indeed the g. c. d. of 96 and 356.

Factoring into primes, we have

$$96 = 2^5 \times 3 = (2^2) \times 2^3 \times 3$$

$$356 = (2^2) \times 89$$

So the g. c. d. is 4 and apparently the above process works. We should ask why. It appears on the surface to be a hit and miss method which just happends to work. To show that the use of this method in finding the g. c. d. is justified is a rather complicated and involved process. It would not suit our purpose to present it here. Such a justification may be found in the appendix.

So to find the g. c. d. of two numbers we may use the process called the Euclidean Algorithm. For convenience, a summary of the steps of the process follows.

To find the g. c. d. of two given numbers we

1. Divide the smaller into the larger, obtaining a remainder r_1.

2. Divide r_1 into the divisor of No. 1, obtaining another remainder r_2.

3. Divide this remainder r_2 into r_1, the divisor of the second division to obtain a remainder r_3.

4. Continue this process of dividing each new remainder into the previous remainder (used as a divisor the time before) until we get a zero remainder.

The last non-zero remainder is the g. c. d. of the two original numbers. Note that we are not concerned with any quotients, only the remainders.

Exercises:

1. Find the g. c. d. of the following pairs of numbers by use of the Euclidean Algorithm.

a. (24, 284)

b. (85, 25)

c. (21, 49)

2. Find the g. c. d. of 8, 32, and 60. (Review what the g. c. d. of two numbers is and then extend this same idea to 3 numbers).

Reducing Fractions

The g. c. d. is used most often in arithmetic in reducing fractions. What we need to find to be able to reduce a fraction is a number which is a common factor of the numerator and denominator and preferably the largest common factor. Generally, the factorization method of finding the g. c. d. is most profitable in this light. For example:

Reduce the fraction $\frac{126}{84}$ to lowest terms.

Factoring, we have

$$126 = 2 \times 3^2 \times 7$$

$$84 = 2^2 \times 3 \times 7$$

Then locating the g. c. d., we can show

$$126 = (2 \times 3 \times 7) \times 3$$

$$84 = (2 \times 3 \times 7) \times 2$$

$$\text{So } \frac{126}{84} = \left(\frac{2 \times 3 \times 7}{2 \times 3 \times 7}\right) \times \frac{3}{2} = (1) \times \left(\frac{3}{2}\right) = \frac{3}{2} .$$

It is obvious if we write it out this way that the g. c. d. has been factored out of both the numerator and denominator giving a fraction equal to 1.

<u>Relationship between g. c. d. and l. c. m.</u>

There is an interesting relationship between the g. c. d and l. c. m. of two numbers which has not yet been made obvious. An example should help us see this relationship; consider 84 and 126.

Find the l. c. m.

$$84 = 2^2 \times 3 \times 7$$

$$126 = 2 \times 3^2 \times 7$$

The l. c. m. of 84 and 126 is $2^2 \times 3^2 \times 7 = 252$

Find the g. c. d.

$$84 = (2 \times 3 \times 7) \times 2$$

$$126 = (2 \times 3 \times 7) \times 3$$

So g. c. d. of 84 and 126 = 2 x 3 x 7 = 42

Consider for a moment the product of 126 and 84.

$$126 \times 84 = 10484.$$

Suppose we factor the product.

$$10484 = 2^3 \times 3^3 \times 7^2$$

and rearrange the factors as

$$10484 = 2^3 \times 3^3 \times 7^2 = (2 \times 3 \times 7) \times (2^2 \times 3^2 \times 7).$$

$$= 42 \times 252$$

After we multiply what do we have on the right? The g. c. d. of 84 and 126 times the l. c. m. of 84 and 126. Let's try the pattern again. Consider 75 and 30.

Factoring:

$$75 = 2 \times 5^2$$

$$30 = 2 \times 3 \times 5$$

For l. c. m., we have

$$2 \times 3 \times 5^2 = 150$$

For g. c. d. we have

$$3 \times 5 = 15$$

The product of 75 and 30 is

$$75 \times 30 = 2250 = 2 \times 3^2 \times 5^3 = (3 \times 5) \times (2 \times 3 \times 5^2)$$

$$= 15 \times 150 .$$

Again the product of the two numbers is equal to the product of the g. c. d. and l. c. m. of the two numbers. This is a general pattern which occurs for any two numbers, i. e., for any two whole non-zero numbers a and b, the following equation holds:

$$a \times b = (\text{g. c. d. of } a \text{ and } b) \times (\text{l. c. m. of } a \text{ and } b).$$

Exercises:

1. For each of the following pairs of numbers, show that $a \times b =$ (l. c. m. of a and b) x (g. c. d. of a and b)

   a. 24 and 284

   b. 21 and 49

   c. 140 and 350

2. Considering the above exercises, suppose you are given the l. c. m. of two numbers, how could you find the g. c. d. by a new method not mentioned previously? Try your idea for 200 and 440 which have an l. c. m. of 88, 000.

What about a case in which you are given the g. c. d. and want to find the l. c. m. by a new method?

Unit 4 ENRICHMENT TOPICS FROM NUMBER THEORY

4.1: Why Use Number Theory For Enrichment ?

As has been mentioned earlier, number theory is an excellent area from which to take enrichment topics for use in elementary school mathematics. For one thing, as you have seen, many problems in this area can be stated very simply and in terms of the simplest numbers, the whole numbers. Also, the history of number theory is one of the richest and longest of any of the branches of mathematics and contains many fascinating stories. For example, number theory really began as a mystical, religious study of numbers. Numbers were assigned magical powers and were considered things that actually existed rather than constructs of the mind. There was actually a secret society devoted to the study of numbers the members of which swore an oath not to reveal their secrets or teaching about numbers. Numbers were named according to what properties they were thought to possess. For example, there were friendly numbers, perfect numbers, abundant numbers, and defective numbers, square numbers and triangular numbers, all with certain properties to be discussed later.

Number theory can also give youngsters a look at the way great mathematicians in the past have worked, in a context the youngsters can understand. There are many patterns which occur in simple addition and multiplication of whole numbers that youngsters can recognize. By

observing these patterns, the elementary student can "discover" on his own and feel that he is really doing mathematics rather than learning rules and memorizing algorithms. Topics from number theory which can be understood by youngsters should be presented to them for their enjoyment and mathematical development. A few such enrichment topics are presented in the following material. This material is suggested not only as enrichment material but also to give the reader a longer and deeper look at the theory of numbers.

We have thus far barely scratched the surface of this area of mathematics. Now we will dig a little deeper.

4.2: Ancient Number Theory

Some comments have been made previously about ancient secret orders which had as their purpose the study of numbers. One such group was the Greek Pythagoreans. This group was formed by the Greek mathematician Pythagoras (c. 550 B. C.) whose name you may have heard mentioned in relation to a famous theorem in geometry. Pythagoras was a pupil of Thales, who is thought to have done the first work in number theory. It was a general practice among members of the society to attribute all credit for each new discovery to Pythagoras, himself, so we cannot be sure about hi ontributions, but it is thought they were very great. The famous philosopher, Plato, was a student of the Pythagoreans.

The Pythagoreans attributed mystical powers and human characteristics to many numbers. The even numbers were thought to be soluble, feminine and pertaining to the earthly, and odd numbers were regarded as indissoluble, masculine, and of celestial nature. One (1) stood for reason, two for opinion, four for justice because it was the first product of equals (4 = 2 x 2), and five suggested marriage, the union of the first masculine and feminine numbers (5 = 3 + 2). One (1) was not considered an odd number, but rather as the source of all numbers, since 2 = 1 + 1 , 3= 1 + 1 + 1, 4 = 1 + 1 + 1 + 1, etc.

There is a striking similarity to these attributes of number in ancient Chinese mythology. Here the odd numbers symbolized white, day, heat, sun, fire, and the even numbers symbolized black, night, cold, matter, water, earth. Magical powers were also attributed to numbers in this mythology.

In Judaeo-Christian traditions there are certain numbers recurring often. Forty days and forty nights of rain, Moses conferred with Jehovah for forty days and forty nights, and the children of Israel wandered forty years in the wilderness. There were the seven deadly sins, the seven virtues, the seven spirits of God, seven joys of the Virgin Mary, and seven devils cast out of Magdalen.

The Babylonians preferred sixty and their gods were assiciated with the numbers up to sixty, the number indicating the rank of the god. Also,

their number system was based on sixty in a manner similar to the way ours is based on ten.

Numbers related in a certain way

If we consider the sets of divisors of the two numbers 220 and 284 we notice something very peculiar. The sets of divisors of each are (excluding the numbers themselves)

of 220, {1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110 }

of 284, {1, 2, 4, 71, 142} .

If we add up the divisors of each we get an interesting result.

$$1 + 2 + 3 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$$

$$1 + 2 + 4 + 71 + 142 = 220 .$$

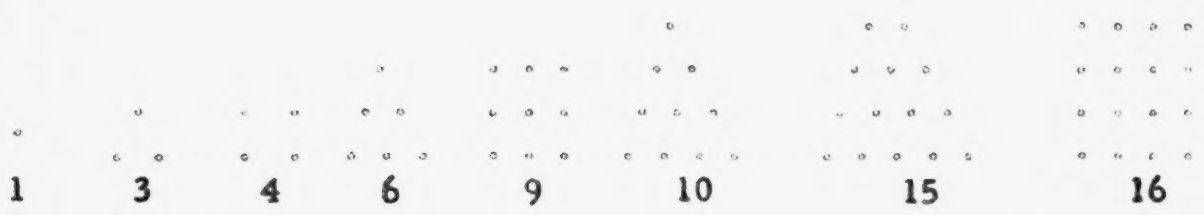
The divisors of each add up to the other !! The Pythagoreans knew of such pairs of numbers (in fact, they knew of this pair) and called them friendly or amicable numbers. The Hindus also knew of them, possibly before the Pythagoreans, and a good omen was attached to such numbers by them. Almost a hundred pairs of friendly numbers are known today (220 and 228 is the smallest pair) but nothing has been proved about how many there actually are of them. Another pair is 1184 and 1210. Can you prove it?

Numbers were also classified as to how they compared with the sum of their proper divisors (that is those divisors which are less than the

given number). For example, the set of proper divisors of 12 is $\{1, 2, 3, 4, 6\}$. The sum of these divisors is 16. The number 12 is called a defective (or deficient) number since it is less than the sum of its proper divisors. Consider the set of divisors of 14, less than 14, $\{1, 2, 7\}$. Their sum is 10. Fourteen is greater than 10, so 14 was called excessive. Now the question arises: Is there a number whose divisors less than the number add up to the number itself? It was known by the Hindus and Hebrews that there are such numbers and these were called perfect numbers. The smallest one is 6. The set divisors of 6 which are less than 6 is $\{1, 2, 3\}$ and $1 + 2 + 3 = 6$. The number 8128 is also a perfect number. Very few such numbers are known today (20 were known in 1961), and it is not known whether an infinite number of them exist or not. Euclid, whose name seems to crop up everywhere in mathematics, discovered a formula which gives even perfect numbers. One interesting fact is that all perfect numbers discovered to date are even. It is not known whether odd perfect numbers exist or not. It is known that any even perfect number must end in 6 or 8.

Figurate numbers

In early Greek days, notably by the Pythagoreans, numbers were recorded by dots. These dots were arranged in arrays which suggested names for the numbers and also allowed properties of the numbers to be derived from the geometric configurations. Consider the following arrays representing the numbers indicated.



From The configurations, numbers such as 1, 3, 6, 10, and 15 were called triangular numbers and numbers such as 1, 4, 9, and 16 were called square numbers. (One (1) was considered to be both a square and a triangular number.) We might note some interesting facts about these numbers. Consider first the triangular numbers. Note that

$1 = 1$

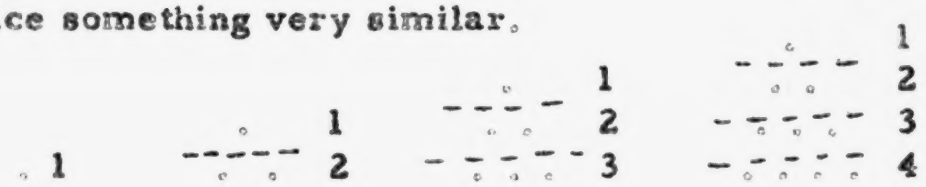
$3 = 1 + 2$

$6 = 1 + 2 + 3$

$10 = 1 + 2 + 3 + 4$

$15 = 1 + 2 + 3 + 4 + 5$

Do you see the pattern ? What would be the next triangular number after 15 ? Is it 21? Is the next one after 21, 28? Think of the joy of accomplishment an elementary student could experience in discovering this for himself. If we look at the rows of the geometrical arrays for these numbers, we notice something very similar.



Thought Exercise

The first triangular number is $1 = 1$, the second is $3 = 1 + 2$, the

third is $6 = 1 + 2 + 3$, and the fourth is $10 = 1 + 2 + 3 + 4$. What would you guess the seventh triangular number to be? The tenth? Try to generalize this. What would the nth triangular number be in terms of n?

Now let's look at the square numbers.

$$1 = 1 \cdot 1$$
$$4 = 2 \cdot 2 = 2^2$$
$$9 = 3 \cdot 3 = 3^2$$
$$16 = 4 \cdot 4 = 4^2$$

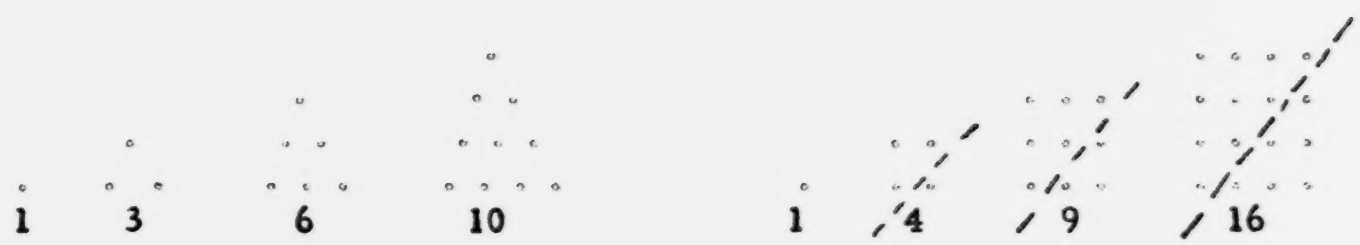
Do you see the pattern? What is the next square number after 16? Is it 25?

Thought Exercise

Notice that the first square number is 1, the second is 4, the third is 9 and the fourth is 16. What do you suppose the seventh one is? Check your answer with a diagram. What do you suppose the tenth one is ? The twenty-fifth? Generalize and tell what the nth one would be in terms of n.

Above we notice that the square number 4 was equal 2^2. We say 2^2 in words as '2 squared'. Also we say 3^2 as "3 squared". In general, we say n^2 as "n squared". What do you suppose is the origin of this term?

There is an interesting relationship between triangular and square numbers which we can discover by looking at the array of dots representing them.



Notice the lines that have been drawn in the square numbers. What has been done with each square number? It has been divided into two <u>triangular numbers</u> !! That is,

$1 = 1$

$4 = 1 + 3$

$9 = 3 + 6$

$16 = 6 + 10$.

In fact the second square number has been divided into the first and second triangular numbers, the third square number has been divided into the second and third triangular numbers, and the fourth square number has been divided into the third and fourth triangular numbers. Might this be true in general? The reader might try his hand at proving it. It is true, though it will not be shown here, as it would necessitate a knowledge of algebra not assumed in this material. Think of the motivation that discovering such patterns would give to a youngster, urging him to press on to try other triangular and square numbers for other patterns. Elementary number theory is permeated with such possibilities for discovery.

<u>More on primes</u>

We have previously discussed prime numbers in a general way.

We will now look at some specific relationships. Consider the pairs of primes 3 and 5, 5 and 7, 11 and 13. The numbers of each pair differ by two. Primes of this type are called twin primes and many have been known since antiquity. It is not known whether there is an infinite number of such pairs or not though it has been known since 1919 that these become rarer and rarer as numbers increase. As has been mentioned earlier, Euclid proved that the number of primes is infinite and this may suggest that many questions about prime number pairs or triples, etc. could be answered in the same manner, but this has not yet proved fruitful in the case of twin primes.

Exercises:

1. Determine whether the following numbers are perfect, defective, or excessive.

   a. 28
   b. 25
   c. 36
   d. 496
   e. 225
   f. 144

2. Show that 96 and 115 are not friendly numbers.

3. What is the ninth triangular number? Show it as a triangular array. What is the twelth triangular number?

4. What is the arithmetical difference between the second and third triangular numbers ? The third and fourth triangular numbers. The

fifth and sixth? The twenty-fifth and twenty-sixth? The nth and the n + 1st (the number following the nth)?

5. What is the ninth square number? The twelth? The twenty-fifth?

6. Of what two triangular numbers is the eighth square number (64) the sum? Of what two triangular numbers is the ninth square number the sum? The twenty-fifth square number? The nth?

7. What is the arithmetical difference between the first and second square numbers? The second and third square numbers? The third and fourth? What is the pattern? Try and generalize this.

8. Determine which of the following are square numbers.

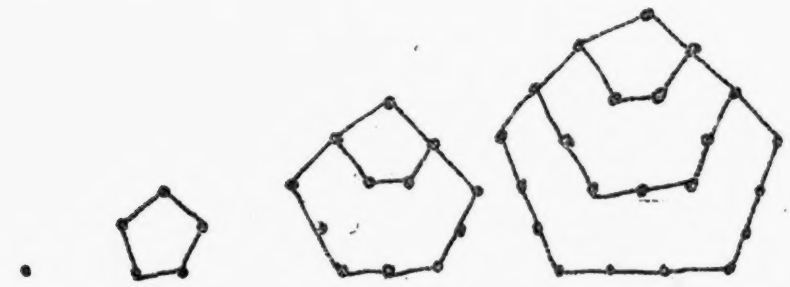
a. 15
b. 36
c. 121
d. 143
e. 144

9. Determine which of the following are triangular numbers.

a. 28
b. 48
c. 45
d. 78

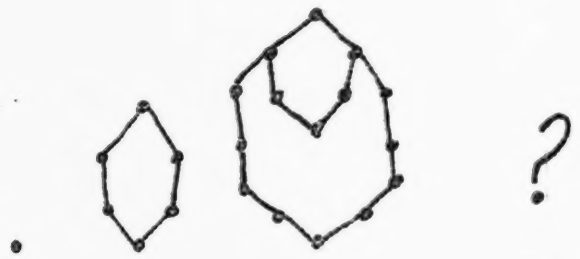
10. The Pythagoreans also called certain numbers pentagonal and hexagonal. For example

some pentagonal numbers are



1 5 12 22

some hexagonal numbers are



1 6 15 28

Lines have been drawn in to help you see the shape and how the figures are formed. Try your hand at discovering patterns. What are the arithmetic differences between successive numbers of each type? Is there any relation between pentagonal and hexagonal numbers? Between these and square and triangular numbers?

4.3 Some Famous Theorems and Conjectures

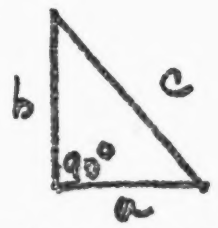
One conclusion about mathematics to which most students seem to come sometime in their elementary training is that mathematics is a fixed and unchanging body of knowledge in which all problems are solved and no questions still unanswered. Many times this serves to leave them with the feeling that mathematics is stagnant and unrewarding to study. All too often we perpetuate this misconception by our emphasis on rules which must be accepted without question and drills which must be carried out. We must try to do just the opposite. It is necessary that students leave us with the impression that mathematics is a vital growing subject, that there are many unsolved problems and new areas to explore. The theory of numbers can offer some concrete examples.

There are many famous, unsolved problems in the theory of numbers. Most of these go by the name of a conjecture, usually associated with the mathematician who first formulated them. One of the most famous of these is called a theorem, though it really is still a conjecture as no proof of it has ever been recorded. It comes to us from a famous French mathematician of the seventeenth century, Pierre de Fermat. This man could have been credited with important discoveries in many fields of mathematics. He might have ranked with Descartes in analytic geometry, with Newton and Leibniz in calculus, and with Pascal in probability. Unfortunately, Fermat seems to have had very little interest in publishing his results so the aforementioned gentlemen are credited above him in the listed fields. This disinterest of his in publishing beings us to one of the most famous unsolved problems in number theory, sometimes called Fermat's last theorem.

The story of Fermat's last theorem really goes back to the ancient Greeks and to one in particular, Diophantus of Alexandria (c. 275 A. D.). Diophantus wrote a work called the Arithmetica which brought together the algebraic knowledge of the Greeks. In this work, there was a discussion of a theorem well-known to anyone who has studied plane goemetry in school. This theorem is the so-called Pythagorean Theorem. What is says is this:

> Given any right triangle whose sides are of length a, b, and c, c being the length of the side opposite the right angle, then the following relation holds for a, b, and c.

$a^2 + b^2 = c^2$



In the Arithmetica there was a discussion of triples of integers a, b, and c which satisfied the above relation. Two examples of such triples are: a = 3, b = 4, and c = 5 and a = 5, b = 12, and c = 13. We can show that these are triples satisfying the above relation as follows.

$3^2 + 4^2 = 9 + 16 = 25 = 5^2$

so $3^2 + 4^2 = 5^2$

and $5^2 + 12^2 = 25 + 144 = 169 = 13^2$.

So $5^2 + 12^2 = 13^2$.

Obviously enough, such triples of numbers are called Pythagorean triples. Rules were given in the Arithmetica for determining Pythagorean triples.

Now let's return to our hero (or villain, whichever the case may be), Fermat. He had obtained a translation of Diophantus' work and was very intrigued by it. He studied the Pythagorean triples and tried to make generalizations. Out of this came his last theorem. In the margin of his copy of Diophantus, he wrote (this is paraphrased)

"It is impossible to have 3 integers

a, b, and c such that

$a^3 + b^3 = c^3$

or $a^4 + b^4 = c^4$

Or, in general, for any n greater than 2 it is impossible to have three integers a, b, and c such that

$a^n + b^n = c^n$.

I have discovered a truly wonderful proof for this but the margin is too small to hold it."

He never published his proof. Mathematicians, both brilliant and not so brilliant, have been trying ever since then to prove or disprove his conjecture. It has been proved for some particular values of n, but as we have seen this does not suffice to prove it in general. In fact, it has been proved for some values of n up to 250,000,000.

Exercise

Another set of Pythagorean numbers is 6, 8 and 10. We can check this by

$$6^2 + 8^2 = 36 + 64 = 100 = 10^2$$

Hence $6^2 + 8^2 = 10^2$.

What relationship can you see between this set of triples and the set 3, 4, and 5? Try the set 9, 12, and 15. Is this a Pythagorean triple? What would be another Pythagorean triple? Can you generalize? Try your generalization on another set of Pythagorean numbers, say 5, 12 and 13. What does this suggest about the number of Pythagorean triples? (How many are there?)

Another famous conjecture is that of Goldbach, a Prussian mathematician of the eighteenth century. Goldbach's conjecture deals with prime numbers. We can easily develop this conjecture by observing the following:

$2 = 1 + 1$

$4 = 1 + 3$

$6 = 3 + 3$

$8 = 3 + 5$

$10 = 3 + 7$

$12 = 5 + 7$

$14 = 7 + 7$

$\vdots$

On the left, we have all even numbers through 14 (excluding 0). What do we have on the right? What kind of number is each number in the sums? A prime !! We might conjecture that "every even number is the sum of two primes". Such a simple statement should be very easy to prove, after all we know a lot about even numbers and a lot about primes. There is one sour note. Mathematicians have been trying for 200 years to prove it and have not yet succeeded !! It is interesting that it has been verified for all numbers up to 10,000 and some beyond. But it may be wrong for 10,563,264 !! Again, we have an example of a statement which is known to be true in many specific cases but has not been accepted because no one has proved it in general. Such is the nature of mathematics

Exercise

Write all even numbers up to 50 as the sum of two primes. Can any even number be represented as the sum of two primes in more than

one way? Can the sum of two primes ever be an odd number?

We have mentioned in a previous chapter that a formula has been discovered which gives the approximate number of primes less than a given number. Some more specific theorems have been proved about the occurrence of primes in intervals between numbers. For example, in 1845, the French mathematician Bertrand conjectured that between any number and its double there is at least one prime. For example, between 3 and its double, 6, there is the prime 5. Between 10 and 20 there is the prime 13. Between 50 and 100 there is the prime 89. Fifty years after Bertrand's conjecture, the Russian Tchebyshev proved it. In 1911, Bonolis improved on this by giving the formula approximating the number of primes between x and $\frac{3}{2}x$, for any x, that is, between a number and another number half again as large.

Exercise

Find a prime between 15 and its double ; between 20 and its double ; between 75 and its double. Can there be more than one?

We would not need to stop here in our look at famous problems, solved and unsolved, but the list is endless and we shall not go on. If your curiosity has been whetted, then the desired end has been accomplished. It is hoped that you have been stirred to make a deeper search into mathematics in general and the field of number theory in particular. Little more needs to be said except to restate

the conviction that the theory of numbers is a field of mathematics which can make important contributions to the teaching and learning of elementary school mathematics. Its importance and value as a motivational device and as a representation of mathematical beauty and truth should not be overlooked. If you are convinced of that then these units have served their purpose.

Appendix

I. The set of primes is an infinite set.

Proof:

A discussion which can be interpreted as a proof of this statement was given by Euclid over 2000 years ago. The proof given here is similar. It is of the type called an indirect proof. Recall that this method was used in proving that the set of even numbers and the set of odd numbers are disjoint.

The indirect method proceeds as follows. We assume the opposite of what we want to prove and see where it leads us. If such an assumption leads to something false or contradictory then we know the assumption was false and hence its opposite must be true.

Thus we begin the proof that the set of primes, P, is infinite with the assumption that this set is not infinite. This is equivalent to saying that the set is finite. We have not spoken of the relation between finite and infinite sets, but a set is either one or the other. There are no other types in this sense.

Now, one meaning of the statement that the set of primes, P, is finite is that we could make a complete listing of the set, such as,

$$P = p_1, p_2, p_3, \ldots, p_n .$$

This says that $p_1, p_2, p_3, \ldots, p_n$ is a listing of all the prime numbers and that there are only n primes.

Let k be a new number formed by multiplying all the primes together and adding 1. Then

(A) $$k = (p_1 \times p_2 \times p_3 \times \ldots \times p_n) + 1.$$

Now k is larger than any of the primes. We write this as

$$p_1 < k,\ p_2 < k,\ p_3 < k,\ \ldots,\ p_n < k,$$

and say "p_1 is less than k", "p_2 is less than k" and so on. Also k is either prime or composite, since it is neither 0 nor 1. (It is not 0 because it is equal to some number plus 1. It is not 1 since this would be the case only if

$$k = 0 + 1$$

so that

$$p_1 \times p_2 \times \dots \times p_n = 0.$$

This would mean that one (or more) of the primes was 0, but 0 is not a prime.)

Now we will check both of the cases: (1) k is prime and (2) k is composite to see where each leads us.

(1) Suppose k is prime.

We said k is larger than any of the primes in P so k cannot be in P. This contradicts the assumption that P is the set of all primes.

Thus case (1) leads to a contradiction.

(2) Suppose k is composite.

Recall that a composite number has factors other than itself and 1 and in fact has prime factors (by the fundamental theorem of arithmetic). Thus there is a prime which is a factor of k. Since P contains all the primes, this factor of k has to be an element of P. Suppose it is p_1.

Another way to say that p_1 is a factor of k is to say "p_1 divides k" or

$$p_1 \mid k.$$

Since p_1 is a factor of $p_1 \times p_2 \times p_3 \times \dots \times p_n$, p_1 divides $(p_1 \times p_2 \times p_3 \times \dots p_n)$ or

$$p_1 \mid (p_1 \times p_2 \times p_3 \times \dots \times p_n).$$

Thus we have that

(B) $$p_1 \mid k \text{ and } p_1 \mid (p_1 \times p_2 \times p_3 \times \dots \times p_n).$$

We will now use an exercise from section 3.2. This was

if $a \mid b$ and $a \mid c$, then $a \mid (b - c)$.

Since p_1 divides both k and $(p_1 \times p_2 \times p_3 \times \ldots \times p_n)$ it divides their difference.

(C) $$p_1 \mid [k - (p_1 \times p_2 \times p_3 \times \ldots \times p_n)].$$

Recall that

$$k = (p_1 \times p_2 \times p_3 \times \ldots \times p_n) + 1.$$

so

(D) $$k - (p_1 \times p_2 \times p_3 \times \ldots \times p_n) = 1.$$

From equation (D) and statement (C) we have

(E) $$p_1 \mid 1.$$

and this means that $p_1 = 1$ since the only whole number divisor of 1 is 1.

We now have a contradiction. We have that $p_1 = 1$ and p_1 is prime which cannot be since 1 is not prime.

The assumption that led us to this contradiction is that $p_1 \mid k$. Recall that this came from our assumption that k is composite and that it therefore must have a prime factor.

Suppose instead of p_1 we had taken p_2 to be the prime factor of k. Can you see that this would make no difference? In mathematics we deal with generalizations so that we may prove things with one general case rather than many specific cases. In our proof we chose p_1 as the prime that divided k. There was nothing special about p_1 and anything that we have proved about it holds for p_2 or p_3 or any of the primes in P.

Thus no prime in P can be a factor of k, because if we assume that one of them is a factor of k we arrive at the contradiction that the prime must be 1, as in (E).

So if k is composite then there must be some prime which is a factor of it that is not in P.

(3) Summary.

We started with a set of primes, P, that we assumed to contain all the primes. We formed a new number, k, from all these primes by taking their product and adding 1. We saw that k was not in the set P.

If k was prime then P could not be the set of all primes. Also, we saw that if k was composite then it had to have a prime factor that was some prime not in P. In either case P could not be the set of all primes.

Thus the assumption that P contained all primes led us to the conclusion that there was a prime not in P. Obviously, the conclusion contradicts the original assumption. This tells us that this assumption, that P was a finite set, must be false. We also said that P must be either finite or infinite. Thus P must be infinite.

II. Use of the Euclidean Division Algorithm to find the g.c.d. of two numbers.

We will first use the algorithm to find a number associated with 18 and 48. We will then show that this number is the g.c.d. of 18 and 48.

Recall that to find the g.c.d. of two numbers by use of the algorithm we first divide the smaller of the two numbers into the larger. Next we divide the remainder of this division into the divisor of this division. This process of dividing the remainder into the divisor is continued until a zero remainder is obtained. The last non-zero remainder is the desired number.

A. Find the g.c.d. of 18 and 48:

(1)
$$\begin{array}{r} 2 \\ 18\overline{)48} \\ \underline{36} \\ 12 \end{array}$$

(2)
$$\begin{array}{r} 1 \\ 12\overline{)18} \\ \underline{12} \\ 6 \end{array}$$

(3)
$$\begin{array}{r} 2 \\ 6\overline{)12} \\ \underline{12} \\ 0 \end{array}$$

Thus 6 is the number determined by the algorithm.

B. Show that 6 is a common divisor of 18 and 48: For this special case all that is needed is to divide both 18 and 48 by 6. However we shall use a more involved process because it can be made into a general proof of the process for all pairs of numbers.

(4) We know from (3) above that

$$12 = 6 \times 2.$$

This means that $6 \mid 12$.

(5) From (2) we know that

$$18 = (12 \times 1) + 6$$

(6) From (4) we know

$6 \mid (12 \times 1)$. Also we know

$6 \mid 6$.

From a statement (theorem) which we proved on page 53 we know that if $a \mid b$ and $a \mid c$ then $a \mid (b + c)$. Using this, we have $6 \mid (12 \times 1)$ and $6 \mid 6$, so $6 \mid [(12 \times 1) + 6]$ or, since in (5) we have $18 = (12 \times 1) + 6$, $6 \mid 18$.

(7) If $6 \mid 18$ then 6 is a factor of 18 and it is also a factor of 18×2 so $6 \mid (18 \times 2)$.

Using the theorem from page 53 again,

$$6 \mid (18 \times 2) \text{ and } 6 \mid 12$$

so

$$6 \mid [(18 \times 2) + 12].$$

or since

$$(18 \times 2) + 12 = 48$$

$$6 \mid 48.$$

(8) Now we have

$$6 \mid 48 \text{ and } 6 \mid 18.$$

Thus 6 is a common factor of 18 and 48. We have not yet shown that it is the greatest common factor. To do this we need to show that any common factor of 18 and 48 divides 6. (See Definition 14, page 68.)

C. Show that 6 is the greatest common divisor of 18 and 48.

(9) Let c represent any common factor of 18 and 48. Now we want to show that $c \mid 6$.

(10) In (5) we saw that

$$18 = (12 \times 1) + 6 = 12 + 6$$

From this we can get

$$12 = 18 - 6$$

(11) From (1) by the Division Algorithm we can get

$$48 = (18 \times 2) + 12$$

(12) If we substitute for 12 in this equation the expression for 12 found in (10) we get $48 = (18 \times 2) + 18 - 6$.

We can substitute 18×1 for 18 and get

$$48 = (18 \times 2) + (18 \times 1) - 6.$$

Then by the distributive law

$$48 = 18 \times (2 + 1) - 6$$

$$= (18 \times 3) - 6$$

From this equation we can get

$$6 = (18 \times 3) - 48.$$

This is the equation we want because it is in terms of 6, 18, and 48, the numbers in which we are interested.

(13) Let's consider c again. We know that c is a common factor of 18 and 48 so

$$c \mid 18 \text{ and } c \mid 48.$$

(14) Now if $c \mid 18$ then $c \mid (18 \times 3)$.

(15) Also if $c \mid (18 \times 3)$ and $c \mid 48$ then by exercise 3 on page 54,

$$c \mid [(18 \times 3) - 48].$$

(16) From (12) we have

$$6 = (18 \times 3) - 48.$$

Thus

$$c \mid 6.$$

This is what we wanted to show. c represents any common factor of 18 and 48 and we have just shown that $c \mid 6$. But 6 is the largest factor of itself, 6, hence 6 is the g.c.d. of 18 and 48.

We have not really proved anything except that the Euclidean Algorithm can be used to find the g.c.d. of 18 and 48. This is not sufficient to prove that it works in general. To do this we would have to make a proof dealing with general numbers and not specific ones. Many times in mathematics it is instructive to see how a theorem can be proved for one specific case and then use this as a model for a more general proof.

A general proof that the Euclidean Algorithm can be used in all cases is not given here as some tools are needed that we do not have at our disposal. The general proof proceeds very much in the same manner as we have seen for 18 and 48. The Division Algorithm is used as we did in (4), (5) and (11) only in a more general form such as for every pair of positive integers a and b, there are two other positive integers q and r such that $a = bq + r$ where $r < b$.

Page 15

1) Yes
$610 = 2 \times 305$ $324 = 2 \times 162$ $1024 = 2 \times 512$

2) Yes, $4 \times n = 2 \times (2 \times n)$ and $(2 \times n) \in W$.

Yes, $6 \times m = 2 \times (3 \times m)$ and $(3 \times m) \in W$.

Yes, $36 \times k = 2 \times (18 \times k)$ and $(18 \times k) \in W$.

3) There is no whole number k such that $25 = 2 \times k$. For this to be true $k = 12\frac{1}{2}$ which is not a whole number.

Page 16 Thought exercises

$n = 0$,	$(2 \times 0)+1$	1
$n = 1$,	$(2 \times 1)+1$	3
$n = 2$,	$(2 \times 2)+1$	5
$n = 3$,	$(2 \times 3)+1$	7
⋮	⋮	⋮

Apparently we do.

Page 20

1) Let $(2 \times n)+1$ and $(2 \times m)+1$ represent any two odd numbers. Find their sum.

$$(2 \times n+1)+(2 \times m+1) = (2 \times n+2 \times m)+(1+1),$$

by the assoc. and comm. principles.

By the distribution law we get

$$2 \times (n+m)+2,$$

and by applying the distribution law again

$$2 \times (n+m+1)$$

...

which is an even number.

So the sum of two odd numbers $2 \times n \quad 1$ and $2 \times m \quad 1$ is an even number.

Find the product of the two given odd numbers

$$(2 \times n+1) \times (2 \times m+1).$$

This is the algebraic product of two binomial expressions which is equal to

$$(2 \times n) \times (2 \times m)+(2 \times n) \times 1+(2 \times m) \times 1+1$$

or

$$(4 \times n \times m) + (2 \times n) + (2 \times m) + 1.$$

Using the distribution principle on the first 3 terms we have

$$2 \times [(2 \times n \times m) + n + m] + 1$$

which is

$$2 \times (\text{some whole number}) + 1$$

So the product of two odd numbers is odd.

2) 2 x n is an even number and (2 x m)+1 is an odd number. Find their sum.

$(2 \times n) + (2 \times m) + 1 = 2 \times (n+m) + 1$ by the distribution principle.

This is

$2 \times (\text{some whole number}) + 1$

an odd number

3) Using the odd and even numbers from exercise 2 find their product.

$(2 \times n) \times [(2 \times m) + 1] = (2 \times n) \times (2 \times m) + (2 \times n)$

by the distribution principle. Using the associative principle we can get

$2 \times (n \times 2 \times m) + (2 \times n)$

and by the distributive principle

$2 \times [(n \times 2 \times m) + n]$.

and even number.

4) See exercise 1.

Page 22

1) $2n \quad 4 = (2n+3) + 1$ and $2n+4 = (2n+5) - 1$

So $2n+4$ is one more than $2n+3$ and one less than $2n+5$.

2) Likewise, $2n+7 = (2n+6) + 1$ and $2n+7 = (2n+8) - 1$

3) Between $2n+9$ and $2n+11$.

$2n+10$ since $2n+10 = (2n+9)\ 1$ and $2n+10 = (2n+11) - 1$.

Between $2n+16$ and $2n+18$.

$2n+17$ since $2n+17 = (2n+16)\ 1$ and $2n+17 = (2n+18) - 1$.

Between $2n+101$ and $2n+103$.

$2n+102$ since $2n+102 = (2n+101)+1$ and $2n+102 = (2n+103)-1$

Page 26

Thought exercise

0,3,6,9,12,15.
Yes
Yes
Yes, $0 = 3 \times 0$.

Page 28

1) A set of multiples of $5 = \{0,5,10,15\}$, $0 = 5 \times 0$, $5 = 5 \times 1$, $10 = 5 \times 2$, $15 = 5 \times 3$

A set of multiples of $6 = \{0,6,12,18\}$, $0 = 6 \times 0$, $6 = 6 \times 1$, $12 = 6 \times 2$, $18 = 6 \times 3$.

A set of multiples of $7 = \{0,7,14,21\}$, $0 = 7 \times 0$, $7 = 7 \times 1$, $14 = 7 \times 2$, $21 = 7 \times 3$.

2) General elements of the respective sets of multiples are

$3 \times n$, $4 \times m$, $5 \times k$, 6×1, $7 \times p$. (any letters will suffice)

3) 6 is a multiple of 1,2,3, and 6, since $6 = 1 \times 6$ and $6 = 2 \times 3$.
Factors are 1,2,3, and 6. See previous answer

12 is a multiple of 1,2,3,4,6, and 12, since $12 = 1 \times 12$, $12 = 2 \times 6$, and $12 = 3 \times 4$.
Factors are 1,2,3,4,6, and 12.

4) $70 = 2 \times 5 \times 7$
Factors are 1,2,5,7,10,14,35,70.

5) No. Yes. Yes. No.

6)

n	factor of n	reason
5	1,5	$1 \times 5 = 5$
6	1,2,3,6	$1 \times 6 = 6$ $2 \times 3 = 6$
7	1,7	$1 \times 7 = 7$
8	1,2,4,8	$1 \times 8 = 8$ $2 \times 4 = 8$ $2 \times 2 \times 2 = 8$
9	1,3,9	$1 \times 9 = 9$ $3 \times 3 = 9$
10	1,2,5,10	$1 \times 10 = 10$ $2 \times 5 = 10$
0	1,2,3,4....	$1 \times 0 = 0$ $2 \times 0 = 0$ $3 \times 0 = 0$ ⋮

1 is a factor of every whole number
1 x n = n for any whole number, n.

7) List some elements of B,C, and D to show that they are the same as those in A.

Page 39 Thought exercise

$E \cap P = \{2\}$
$E \cap C = \{4,6,8,10,\ldots\}$ (all even numbers except 2)
$O \cap C = \{9,15,21,25,\ldots\}$ (all odd numbers which are not prime)
$O \cap P = \{3,5,7,11,13,17,\ldots\}$ (all odd numbers which are prime)

Page 41

1) $16 = 1 \times 16 = 2 \times 8 = 2 \times 2 \times 4 = 2 \times 2 \times 2 \times 2 = 4 \times 4$
Factors are 1,2,4,8, and 16.

2) $4 = 1 \times 4 = 2 \times 2$
Factors are 1,2, and 4

3) The factorizations of 4 are "contained in" the factorizations of 16.

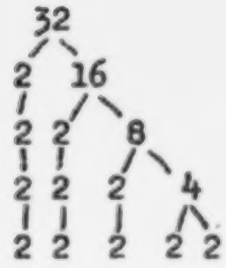
4) Yes, some qualification of the term "factorization" is needed in order to avoid the repeating of ones.

Page 46 a.

1) a) $32 = 2 \times 2 \times 2 \times 2 \times 2$, since

$$\begin{array}{r} 2 \\ 2\overline{)4} \\ 2\overline{)8} \\ 2\overline{)16} \\ 2\overline{)32} \end{array}$$

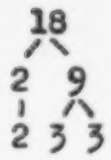
or



b) $18 = 2 \times 3 \times 3$, since

$$\begin{array}{r} 3 \\ 3\overline{)9} \\ 2\overline{)18} \end{array}$$

or



In the same manner
c) $38 = 2 \times 19$
d) $16 = 2 \times 2 \times 2 \times 2$
e) $36 = 2 \times 2 \times 3 \times 3$
f) $272 = 2 \times 2 \times 2 \times 2 \times 17$
g) $45 = 3 \times 3 \times 5$
h) $25 = 2 \times 2$
i) $700 = 2 \times 2 \times 5 \times 5 \times 7$
j) $75 = 3 \times 5 \times 5$

2) a) $32 = 2^5$
b) $18 = 2 \times 3^2$
c) $38 = 2_4 \times 19$
d) $16 = 2_2$
e) $36 = 2^2{}_4 \times 3^2$
f) $272 = 2^4 \times 17$
g) $45 = 3^2_2 \times 5$
h) $25 = 5^2$
i) $700 = 2^2 \times_2 5^2 \times 7$
j) $75 = 3 \times 5^2$

3) $32 = 2^5 = 2 \times 2^4 = 2 \times 16$

$36 = 2^2 \times 3^2 = 2 \times (2 \times 3^2) = 2 \times 18$

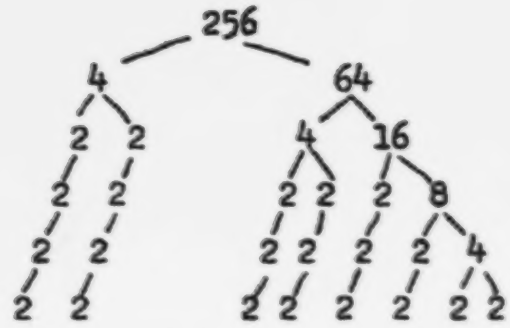
$75 = 3 \times 5^2 = 3 \times 25$

The larger number "contains" the factorization of the smaller.

4) $256 = 2^8$, since

$$\begin{array}{r} 2 \\ 2\overline{)4} \\ 2\overline{)8} \\ 2\overline{)16} \\ 2\overline{)32} \\ 2\overline{)64} \\ 2\overline{)128} \\ 2\overline{)256} \end{array}$$

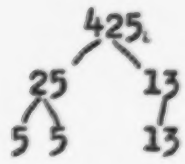
<u>or</u>



$425 = 5^2 \times 13$, since

$$\begin{array}{r} 13 \\ 5\overline{)85} \\ 5\overline{)425} \end{array}$$

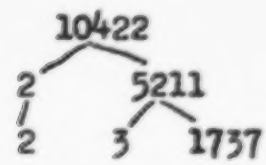
<u>or</u>



$10{,}422 = 2 \times 3 \times 1737$, since

$$\begin{array}{r} 1737 \\ 3\overline{)5211} \\ 2\overline{)10422} \end{array}$$

<u>or</u>



<u>Page 54</u>

1) $2/4$ and $2/8$, $2/12$, $2/16$ since $8 = 2 \times 4$, $12 = 3 \times 4$ and $16 = 4 \times 4$

Proof:

a/b so $b = k \times a$ by definition. P. 48.
Then $b \times c = (b \times a) \times c$ when c is any whole number
So we can get
$b \times c = (k \times a) \times c = (k \times c) \times a$

by the associative and commutative principles.

Then we see that

$$b \times c = (\text{some whole number}) \times a$$

so

$$a \mid (b \times c)$$

2) Numerical examples as above.

Proof

$a \mid (b + c)$ means that from the definition P. 48

$(b + c) = k \times a$

Also

$a \mid b$ means

$b = 1 \times a$.

Now we can get by substituting $1 \times a$ for b

$1 \times a + c = k \times a$

Using some algebra we can get

$c = (k \times a) - (1 \times a)$

and by the distributive principle

$c = k (k - 1) \times a$.

Thus

$c = (\text{some whole number}) \times a$.

(We really should justify that $k - 1$ is a whole number and not negative. This can be done by referring to

$b + c = k \times a$ and $b = 1 \times a$.

Is it not obvious that b c is larger than b? Is it also not obvious that k is larger than 1?)

So

$a \mid c$.

3) Numerical examples as above

$a \mid b$ means $b = k \times a$ and $a \mid c$ means $c = 1 \times a$.

Now

$b - c = (k \times a) - (1 \times a)$.

So

$b - c = (k - 1) \times a$ by the distributive principle.

Then

$b - c = (\text{some whole number}) \times a$ (as above)

And

$a \mid (b - c)$

Page 57

1) $98 = (12 \times 6) + 2$
2) $125 = (25 \times 5) + 0$
3) $49 = (5 \times 9) + 4$

Page 62

1) $9 = 3^2$, $15 = 3 \times 5$
So l.c.m. of 9 & 15 is $3^2 \times 5 = 45$.

$40 = 2^3 \times 5$, $20 = 2^2 \times 5$
So l.c.m. of 40 & 20 is $2^3 \times 5 = 40$.

$60 = 2^2 \times 3 \times 5$, $225 = 3^2 \times 5^2$
So l.c.m. of 60 & 225 is $2^2 \times 3^2 \times 5^2 = 900$.

2) $2 = 2^1$, $5 = 5^1$, $6 = 2 \times 3$
So l.c.m. is $2 \times 5 \times 3 = 30$

3) Using 2 as a factor 5 times and 3 as a factor once gives a common multiple but not the least common multiple.

The least common multiple can also be found by intersection of sets of multiples as on pp. 57-59.

Page 68

1) Set of divisors of 8 = {1,2,4,8}
Set of divisors of 16 = {1,2,4,8,16}
The intersection of these is {1,2,4,8}
g.c.d. is 8
or
$8 = 2^3$, $16 = 2^4$
So the g.c.d. is $2^3 = 8$.

2) Set of divisors of 25 = {1,5,25}
Set of divisors of 45 = {1,3,5,9,15,45}
The intersection of these is {1,5}
So the g.c.d. is 5.
or
$25 = 5^2$, $45 = 3^2 \times 5$
So the g.c.d. is 5.

3) Set of divisors of 108 = {1,2,3,4,6,9,12,18,27,36,54,108}
Set of divisors of 48 = {1,2,3,4,6,8,12,16,24,48}
The intersection of these is {1,2,3,4,6,12}
So the g.c.d. is 12.
or
$108 = 2^2 \times 3^3$, $48 = 2^4 \times 3$
So the g.c.d. is $2^2 \times 3 = 12$.

4) Set of divisors of 54 = {1,2,3,6,9,18,27,54}
Set of divisors of 36 = {1,2,3,4,6,9,12,18,36}
Intersection of these is {1,2,3,6,9,18}
So the g.c.d. is 18

or

$54 = 2 \times 3^3$, $36 = 2^2 \times 3^2$
So the g.c.d. is $2 \times 3^2 = 18$.

Page 71

1)

a)

$$24\overline{)284}\ 11,\ 24,\ 44,\ 24,\ 20 \qquad 20\overline{)24}\ 1,\ 20,\ 4 \qquad 4\overline{)20}\ 5$$

So the g.c.d. is 4.

b)

$$25\overline{)85}\ 3,\ 75,\ 10 \qquad 10\overline{)25}\ 2,\ 20,\ 5 \qquad 5\overline{)10}\ 2$$

So the g.c.d. is 5.

c)

$$21\overline{)49}\ 2,\ 42,\ 7 \qquad 7\overline{)21}\ 3$$

So the g.c.d. is 7.

2) $8 = 2^3$, $32 = 2^5$, $60 = 2^2 \times 3 \times 5$
So the g.c.d. is 2^2.

Page 74

1) a) $24 = 2^3 \times 3$, $284 = 2^2 \times 71$
l.c.m. $= 2^3 \times 3 \times 71 = 1704$
g.c.d. $= 2^2 = 4$
l.c.m. x g.c.d. $= 1704 \times 4 = 6816$
$24 \times 284 = 6816$

b) $21 = 3 \times 7$, $49 = 7^2$
l.c.m. $= 3 \times 7^2 = 147$
g.c.d. $= 7$
l.c.m. x g.c.d. $= 147 \times 7 = 1029$
$21 \times 49 = 1029$.

c) $140 = 2^2 \times 5 \times 7$, $350 = 2 \times 5^2 \times 7$
l.c.m. $= 2^2 \times 5^2 \times 7 = 700$
g.c.d. $= 2 \times 5 \times 7 = 70$

l.c.m. x g.c.d. = 700 x 70 = 49,000

140 x 350 = 49,000

2) $\text{g.c.d.} = \frac{\text{the product of the two numbers}}{\text{l.c.m.}}$

There is an error in the problem: l.c.m. = 2200

$\text{g.c.d.} = \frac{200 \times 440}{2200} = 40$

$\text{l.c.m.} = \frac{\text{the product of the two numbers}}{\text{g.c.d}}$

Page 80 Thought exercise

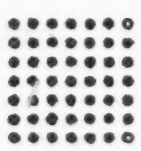
$1+2+3+4+5+6+7 = 28$

$1+2+\ldots+8+9 = 45$

$1+2+\ldots+(n-1)+n$ = the n^{th} triangular number.

Page 81 Thought exercise

$7 \cdot 7 = 49$



(7 by 7 and 49 dots)

$10 \cdot 10 = 100$

$25 \cdot 25 = 625$

$n \cdot n = n^2$

Page 83

1) a) Perfect since $1+2+4+7+14 = 28$

b) Excessive since $1+5 = 6$

c) Defective since $1+2+3+4+6+9+12+18 = 55$

d) Perfect since $1+2+4+8+16+31+62+124 \quad 248 = 496$

e) Excessive since $1+3+5+15+25+45+75 = 169$

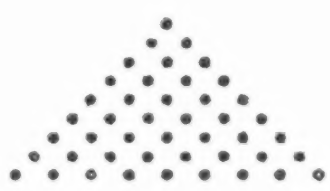
f) Defective since $1+2+3+4+6+8+9+12+16+18+24+36+48+72 = 259$

In each case the proper divisors of the number have been added

2) Factors of 115 are 5 and 23 and $5 \quad 23 \neq 96$

3) $1+2+3+4+5+6+7+8+9 = 45$

$1+2+3+\ldots+11+12 = 78$



4) $6 - 3 = 3$, $10 - 6 = 4$, $21 - 15 = 6$, $26^{th} - 25^{th} = 26$, $n+1^{st} - n^{th} = n+1$.

5) $9^2 = 81$, $12^2 = 144$, $25^2 = 625$.

6) The seventh and eighth, $64 = 28+36$.
The eighth and ninth, $81 = 36+45$.
The twenty-fourth and twenty-fifth,
The n^{th} and $n - 1^{st}$.

7) $4 - 1 = 3$
$9 - 4 = 5$
$16 - 9 = 7$

All the differences are odd numbers.

The n^{th} square number - the $n - 1^{st}$ square number = the n^{th} odd number.

8) b) $36 = 6^2$

c) $121 = 11^2$

e) $144 = 12^2$

9) a) $28 = 1+2+3+4+5+6+7$
c) $45 = 1+2+3+\ldots+7+8+9$
d) $78 = 1+2+3+\ldots+11+12$

10) Differences between successive numbers

Pentagonal

$5 - 1 = 4$
$12 - 5 = 7$
$22 - 12 = 10$

Pattern seems to be a difference of 3 between successive differences.

Hexagonal

$6 - 1 = 5$
$15 - 6 = 9$
$28 - 15 = 13$

Pattern seems to be a difference of 4 between successive differences.

?

Page 88 Exercise

Each number is twice the corresponding number in the first pair.

Yes, since $9^2+12^2 = 81+144 = 225 = 15^2$

The generalization is:

If a,b,c is a Pythagorean triple

then $a \cdot x$, $b \cdot x$, $c \cdot x$ is also a Pythagorean triple (X is any whole number)

10^2 $24^2 = 100$ $576 = 676 = 26^2$

There are an infinite number

Page 89 Exercise (Possible answer)

2 = 1 + 1	16 = 5 + 11	30 = 11 + 19	44 = 7 + 37
4 = 2 + 2	18 = 7 + 11	32 = 3 + 29	46 = 3 + 43
6 = 3 + 3	20 = 7 + 13	34 = 5 + 29	48 = 5 + 43
8 = 3 + 5	22 = 11 + 11	36 = 7 + 29	50 = 7 + 43
10 = 3 + 7	24 = 11 + 13	38 = 7 + 31	
12 = 5 + 7	26 = 13 + 13	40 = 11 + 29	
14 = 7 + 7	28 = 11 + 17	42 = 13 + 29	

Yes

Only if one of the primes is 2.

Page 90 Exercise (Possible answers)

17 is between 15 and 30
23 is between 20 and 40
79 is between 75 and 150

Definitely.